Toric Topology

Victor M. Buchstaber

Taras E. Panov

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina Street 8, 119991 Moscow, Russia
E-mail address: `buchstab@mi.ras.ru`

Department of Mathematics and Mechanics, Moscow State University, Leninskie Gory, 119991 Moscow, Russia
E-mail address: `tpanov@mech.math.msu.su`

Preliminary contents

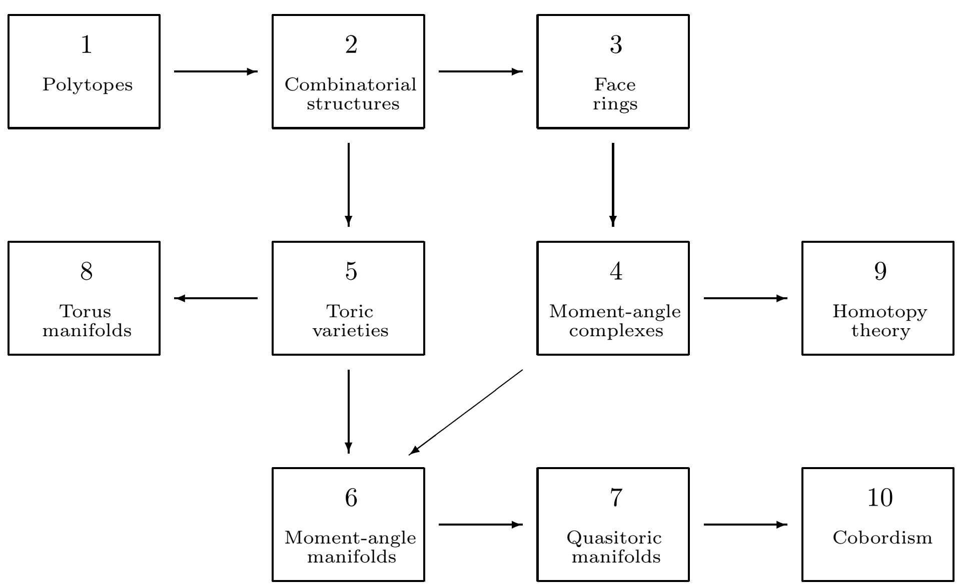


Figure 0.1. Chapter dependence scheme.

CHAPTER 3

Commutative and homological algebra of face rings

The subject of this chapter is the algebraic theory of face rings (also known as Stanley–Reisner rings) of simplicial complexes, and their generalisations to simplicial posets.

With the appearance of the face rings in the beginning of the 1970s in the work of Reisner and Stanley many combinatorial problems were translated into the language of commutative algebra, which paved the way for their solution using the extensive machinery of algebraic and homological methods. Algebraic tools used for attacking combinatorial problems included regular sequences, Cohen–Macaulay and Gorenstein rings, Tor-algebras, local cohomology, etc. A whole new thriving field appeared on the borders of combinatorics and algebra, which has since become known as *combinatorial commutative algebra*. The basic reference here is Stanley's monograph [**184**].

In this chapter we collect the wealth of algebraic notions and constructions related to face rings. Our choice of material and notation was guided by the topological applications in the later chapters of the book. (This explains the unusual for algebraists even grading in the polynomial rings and their homogeneous quotients, and also the nonpositive homological grading in free resolutions and Tor.) In the first sections we review standard results and constructions of combinatorial commutative algebra, including the Tor-algebras and algebraic Betti numbers of face rings, Cohen–Macaulay and Gorenstein complexes. The later sections contain some more recent developments, including the face rings of simplicial posets, different characterisations of Cohen–Macaulay and Gorenstein simplicial posets in terms of their face rings and h-vectors, and generalisations of the Dehn–Sommerville relations. Although all these algebraic and combinatorial results have a strong topological flavour and were indeed originally motivated by topological constructions, we tried to keep this chapter mostly algebraic and do not require much topological knowledge from the reader here.

The preliminary algebraic material of a more general sort, not directly or exclusively related to the face rings (such as resolutions and the functor Tor, and Cohen–Macaulay rings) is collected in Appendices in the end of the book.

Alongside with the above mentioned monograph by Stanley [**184**], an extensive survey of Cohen–Macaulay rings by Bruns and Herzog [**32**] and a more recent monograph [**141**] by Miller and Strurmfels may be recommended for a deeper study of algebraic methods in combinatorics.

Here we use the common notation $\mathbf{k}$ for the ground ring, which is always assumed to be the ring $\mathbb{Z}$ of integers or a field. The former is preferable for topological applications, but the latter is more common in the algebraic literature. We shall

usually refer to $\mathbf{k}$-modules as '$\mathbf{k}$-vector spaces'; in the case $\mathbf{k} = \mathbb{Z}$ the latter means an abelian group.

We assume graded commutativity instead of commutativity; algebras commutative in the standard sense will be those whose nontrivial graded components appear only in even degrees. In particular, the polynomial algebra $\mathbf{k}[v_1, \ldots, v_m]$ (or shortly $\mathbf{k}[m]$) has $\deg v_i = 2$. The exterior algebra $\Lambda[u_1, \ldots, u_m]$ has $\deg u_i = 1$. Given a subset $I = \{i_1, \ldots, i_k\} \subset [m]$ we denote by v_I the square-free monomial $v_{i_1} \cdots v_{i_k}$ in $\mathbf{k}[m]$. We also denote by u_I the exterior monomial $u_{i_1} \cdots u_{i_k}$ where $i_1 < \cdots < i_k$.

3.1. Face rings of simplicial complexes

Definition 3.1.1. The *face ring* (or the *Stanley–Reisner ring*) of a simplicial complex $\mathcal{K}$ on the set $[m]$ is the quotient ring

$$\mathbf{k}[\mathcal{K}] = \mathbf{k}[v_1, \ldots, v_m]/\mathcal{I}_{\mathcal{K}},$$

where $\mathcal{I}_{\mathcal{K}} = (v_I : I \notin \mathcal{K})$ is the homogeneous ideal generated by those monomials v_I for which I is not a simplex of $\mathcal{K}$. The ideal $\mathcal{I}_{\mathcal{K}}$ is referred to as the *Stanley–Reisner ideal* of $\mathcal{K}$.

Example 3.1.2. 1. Let $\mathcal{K}$ be the 2-dimensional simplicial complex shown in Fig. 3.1. Then

$$\mathcal{I}_{\mathcal{K}} = (v_1v_5, v_3v_4, v_1v_2v_3, v_2v_4v_5).$$

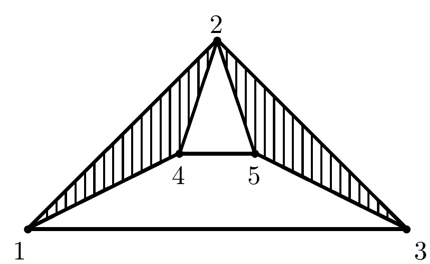


FIGURE 3.1.

2. The face ring $\mathbf{k}[\mathcal{K}]$ is a *quadratic algebra* (that is, the ideal $\mathcal{I}_{\mathcal{K}}$ is generated by quadratic monomials) if and only if $\mathcal{K}$ is a flag complex (an exercise).

3. Let $\mathcal{K}_1 * \mathcal{K}_2$ be the join of $\mathcal{K}_1$ and $\mathcal{K}_2$ (see Construction **??**). Then

$$\mathbf{k}[\mathcal{K}_1 * \mathcal{K}_2] = \mathbf{k}[\mathcal{K}_1] \otimes \mathbf{k}[\mathcal{K}_2].$$

Here and below $\otimes$ denotes the tensor product over $\mathbf{k}$.

We note that $\mathcal{I}_{\mathcal{K}}$ is a *monomial ideal*, and it has a basis consisting of square-free monomials v_I corresponding to the missing faces of $\mathcal{K}$.

Proposition 3.1.3. *Every square-free monomial ideal $\mathcal{I}$ in the polynomial ring is the Stanley–Reisner ideal of a simplicial complex $\mathcal{K}$.*

PROOF. We set

$$\mathcal{K} = \{I \subset [m] : v_I \notin \mathcal{I}\}.$$

Then $\mathcal{K}$ is a simplicial complex and $\mathcal{I} = \mathcal{I}_{\mathcal{K}}$. $\square$

Let P be a simple n-polytope and let $\mathcal{K}_P$ be its nerve complex (see Example **??**). We define the *face ring* $\mathbf{k}[P]$ as the face ring of $\mathcal{K}_P$. Explicitly,

$$\mathbf{k}[P] = \mathbf{k}[v_1, \ldots, v_m]/\mathcal{I}_P,$$

where $\mathcal{I}_P$ is the ideal generated by those square-free monomials $v_{i_1}v_{i_2}\cdots v_{i_s}$ whose corresponding facets intersect trivially, $F_{i_1} \cap \cdots \cap F_{i_s} = \varnothing$.

Example 3.1.4. 1. Let P be an n-simplex (viewed as a simple polytope). Then

$$\mathbf{k}[P] = \mathbf{k}[v_1, \ldots, v_{n+1}]/(v_1v_2\cdots v_{n+1}).$$

2. Let P be a 3-cube I^3. Then

$$\mathbf{k}[P] = \mathbf{k}[v_1, v_2, \ldots, v_6]/(v_1v_4, v_2v_5, v_3v_6).$$

3. Let P be an m-gon, $m \geqslant 4$. Then

$$\mathcal{I}_P = (v_iv_j \colon i - j \neq 0, \pm 1 \mod m).$$

4. Given two simple polytopes P_1 and P_2, we have

$$\mathbf{k}[P_1 \times P_2] = \mathbf{k}[P_1] \otimes \mathbf{k}[P_2].$$

Proposition 3.1.5. *Let $\varphi\colon \mathcal{K}_1 \to \mathcal{K}_2$ be a simplicial map between simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$ on the vertex sets $[m_1]$ and $[m_2]$ respectively. Define the map $\varphi^*\colon \mathbf{k}[w_1, \ldots, w_{m_2}] \to \mathbf{k}[v_1, \ldots, v_{m_1}]$ by*

$$\varphi^*(w_j) = \sum_{i \in \varphi^{-1}(j)} v_i.$$

Then φ^* descends to a homomorphism $\mathbf{k}[\mathcal{K}_2] \to \mathbf{k}[\mathcal{K}_1]$, which we continue to denote φ^*.

Proof. We need to check that $\varphi^*(\mathcal{I}_{\mathcal{K}_2}) \subset \mathcal{I}_{\mathcal{K}_1}$. Suppose $J = \{j_1, \ldots, j_s\} \subset [m_2]$ is not a simplex of $\mathcal{K}_2$. We have

$$\varphi^*(w_{j_1} \ldots w_{j_s}) = \sum_{i_1 \in \varphi^{-1}(j_1), \ldots, i_s \in \varphi^{-1}(j_s)} v_{i_1} \ldots v_{i_s}.$$

We claim that the right hand side above belongs to $\mathcal{I}_{\mathcal{K}_1}$, i.e. for any monomial $v_{i_1} \ldots v_{i_s}$ in the right hand side the set $I = \{i_1, \ldots, i_s\}$ is not a simplex of $\mathcal{K}_1$. Indeed, otherwise we would have $\varphi(I) = J \in \mathcal{K}_2$ by the definition of a simplicial map, which contradicts the assumption. $\square$

Example 3.1.6. The face ring of the barycentric subdivision $\mathcal{K}'$ of $\mathcal{K}$ is

$$\mathbf{k}[\mathcal{K}'] = \mathbf{k}[b_I \colon I \in \mathcal{K} \setminus \varnothing]/\mathcal{I}_{\mathcal{K}'},$$

where b_I is the polynomial generator of degree 2 corresponding to a nonempty simplex $I \in \mathcal{K}$, and $\mathcal{I}_{\mathcal{K}'}$ is generated by quadratic monomials b_Ib_J for which $I \not\subset J$ and $J \not\subset I$. The simplicial map $\nabla\colon \mathcal{K}' \to \mathcal{K}$ from Example **??** induces a map ∇^* of the face ring, given on the generators $v_j \in \mathbf{k}[\mathcal{K}]$ by

$$\nabla^*(v_j) = \sum_{I \in \mathcal{K} \colon \min I = j} b_I.$$

Example 3.1.7. The nondegenerate map $\mathcal{K}' \to \Delta^{n-1}$ from Example **??** induces the following map of the corresponding face rings:

$$\begin{aligned} \mathbf{k}[v_1, \ldots, v_n] &\longrightarrow \mathbf{k}[\mathcal{K}'] \\ v_i &\longmapsto \sum_{|I|=i} b_I. \end{aligned}$$

This defines a canonical $\mathbf{k}[v_1, \ldots, v_n]$-module structure on $\mathbf{k}[\mathcal{K}']$.

An important tool arising from the functoriality of the face ring is the restriction homomorphism. For any simplex $I \in \mathcal{K}$, the corresponding full subcomplex $\mathcal{K}_I$ is $\Delta^{|I|-1}$ and $\mathbf{k}[\mathcal{K}_I]$ is the polynomial ring $\mathbf{k}[v_i \colon i \in I]$ on $|I|$ generators. The inclusion $\mathcal{K}_I \subset \mathcal{K}$ induces the *restriction homomorphism*

$$s_I \colon \mathbf{k}[\mathcal{K}] \to \mathbf{k}[v_i \colon i \in I],$$

which maps v_i to zero whenever $i \notin I$.

The following simple proposition will be used in several algebraic and topological arguments of the later chapters.

Proposition 3.1.8. *The direct sum*

$$s = \bigoplus_{I \in \mathcal{K}} s_I \colon \mathbf{k}[\mathcal{K}] \longrightarrow \bigoplus_{I \in \mathcal{K}} \mathbf{k}[v_i \colon i \in I]$$

of all restriction maps is a monomorphism.

PROOF. Consider the composite map

$$\mathbf{k}[v_1, \ldots, v_m] \xrightarrow{\ p\ } \mathbf{k}[\mathcal{K}] \xrightarrow{\ s\ } \textstyle\bigoplus_{I \in \mathcal{K}} \mathbf{k}[v_i \colon i \in I]$$

where p is the quotient projection. Assume that $s\,p(Q) = 0$ where $Q = Q(v_1, \ldots, v_m)$ is a polynomial. Then for any monomial $v_{i_1}^{\alpha_1} \cdots v_{i_k}^{\alpha_k}$ which enters Q with a nonzero coefficient we have $I = \{i_1, \ldots, i_k\} \notin \mathcal{K}$ (as otherwise the Ith component of the image under $s\,p$ is nonzero). Therefore, $p(Q) = 0$ and s is injective. $\square$

Proposition 3.1.9. *The face ring $\mathbf{k}[\mathcal{K}]$ has the $\mathbf{k}$-vector space basis consisting of monomials $v_{j_1}^{\alpha_1} \cdots v_{j_k}^{\alpha_k}$ where $\alpha_i > 0$ and $\{j_1, \ldots, j_k\} \in \mathcal{K}$.*

PROOF. Indeed, the polynomial algebra $\mathbf{k}[m]$ has the $\mathbf{k}$-vector space basis consisting of all monomials $v_{j_1}^{\alpha_1} \cdots v_{j_k}^{\alpha_k}$, and such a monomial maps to zero under the projection $\mathbf{k}[m] \to \mathbf{k}[\mathcal{K}]$ precisely when $\{j_1, \ldots, j_k\} \notin \mathcal{K}$. $\square$

Recall that the Poincaré series of a nonnegatively graded $\mathbf{k}$-vector space $V = \bigoplus_{i=0}^{\infty} V^i$ is given by $F(V; \lambda) = \sum_{i=0}^{\infty} (\dim_{\mathbf{k}} V^i) \lambda^i$. Since $\mathbf{k}[\mathcal{K}]$ is graded by even numbers, its Poincaré series is even.

Theorem 3.1.10 (Stanley)**.** *Let $\mathcal{K}$ be an $(n-1)$-dimensional simplicial complex with the f-vector $(f_0, \ldots, f_{n-1})$ and the h-vector $(h_0, \ldots, h_n)$. Then the Poincaré series of the face ring $\mathbf{k}[\mathcal{K}]$ is*

$$F\big(\mathbf{k}[\mathcal{K}]; \lambda\big) = \sum_{k=0}^{n} f_{k-1} \left(\frac{\lambda^2}{1 - \lambda^2} \right)^k = \frac{h_0 + h_1 \lambda^2 + \cdots + h_n \lambda^{2n}}{(1 - \lambda^2)^n}.$$

PROOF. By Proposition 3.1.9, a $(k-1)$-dimensional simplex $\{i_1,\dots,i_k\}\in\mathcal{K}$ contributes a summand $\frac{\lambda^{2k}}{(1-\lambda^2)^k}$ to the Poincaré series of $\mathcal{K}$ (this summand is just the Poincaré series of the subspace generated by monomials $v_{i_1}^{\alpha_1}\cdots v_{i_k}^{\alpha_k}$ with positive exponents α_i). This proves the first identity, and the second follows from (**??**). $\square$

Example 3.1.11. 1. Let $\mathcal{K}=\Delta^{n-1}$. Then $f_i=\binom{n}{i+1}$ for $-1\leqslant i\leqslant n-1$, $h_0=1$ and $h_i=0$ for $i>0$. Since every subset of $[n]$ is a simplex of Δ^{n-1}, we have $\mathbf{k}[\Delta^{n-1}]=\mathbf{k}[v_1,\dots,v_n]$ and $F(\mathbf{k}[\Delta^{n-1}];\lambda)=(1-\lambda^2)^{-n}$.

2. Let $\mathcal{K}=\partial\Delta^n$ be the boundary of an n-simplex. Then $h_i=1$ for $i=0,1,\dots,n$, and $\mathbf{k}[\partial\Delta^n]=\mathbf{k}[v_1,\dots,v_{n+1}]/(v_1v_2\cdots v_{n+1})$. By Theorem 3.1.10,

$$F\bigl(\mathbf{k}[\partial\Delta^n];\lambda\bigr)=\frac{1+\lambda^2+\dots+\lambda^{2n}}{(1-\lambda^2)^n}.$$

The affine algebraic variety corresponding to the commutative finitely generated $\mathbf{k}$-algebra $\mathbf{k}[\mathcal{K}]=\mathbf{k}[m]/\mathcal{I}_{\mathcal{K}}$ (i.e. the set of common zeros of elements of $\mathcal{I}_{\mathcal{K}}$, viewed as algebraic functions on $\mathbf{k}^m$) can be easily identified as follows.

Proposition 3.1.12. *The affine algebraic variety corresponding to* $\mathbf{k}[\mathcal{K}]$ *is given by*

$$X(\mathcal{K})=\bigcup_{I\in\mathcal{K}}S_I,$$

where $S_I=\mathbf{k}\langle\boldsymbol{e}_i\colon i\in I\rangle$ *is the coordinate subspace in* $\mathbf{k}^m$ *spanned by the set of standard basis vectors corresponding to* I*.*

PROOF. The statement obviously holds in the case $\mathcal{K}=\Delta^{m-1}$. So we assume $\mathcal{K}\neq\Delta^{m-1}$. We shall use the following notation from Section **??**: $\widehat{I}=[m]\setminus I$, the complement of $I\subset[m]$, and $\widehat{\mathcal{K}}=\{\widehat{I}\in[m]\colon I\notin\mathcal{K}\}$, the dual complex of $\mathcal{K}$. Given a point $\boldsymbol{z}=(z_1,\dots,z_m)\in\mathbf{k}^m$, we denote by

$$\omega(\boldsymbol{z})=\{i\colon z_i=0\}\subset[m],$$

the set of zero coordinates of $\boldsymbol{z}$.

By the definition of the algebraic variety $X(\mathcal{K})$ corresponding to $\mathbf{k}[\mathcal{K}]$,

$$\begin{aligned}X(\mathcal{K})=\bigcap_{J\notin\mathcal{K}}\bigcup_{j\in J}\{\boldsymbol{z}\colon z_j=0\}&=\bigcap_{J\notin\mathcal{K}}\bigl\{\boldsymbol{z}\colon\omega(\boldsymbol{z})\cap J\neq\varnothing\bigr\}\\&=\bigcap_{\widehat{J}\in\widehat{\mathcal{K}}}\bigl\{\boldsymbol{z}\colon\omega(\boldsymbol{z})\not\subset\widehat{J}\bigr\}=\bigl\{\boldsymbol{z}\colon\omega(\boldsymbol{z})\notin\widehat{\mathcal{K}}\bigr\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\bigcup_{I\in\mathcal{K}}S_I=\bigcup_{I\in\mathcal{K}}\bigcap_{j\in\widehat{I}}\{\boldsymbol{z}\colon z_j=0\}&=\bigcup_{I\in\mathcal{K}}\bigl\{\boldsymbol{z}\colon\widehat{I}\subset\omega(\boldsymbol{z})\bigr\}\\&=\bigcup_{\widehat{I}\notin\widehat{\mathcal{K}}}\bigl\{\boldsymbol{z}\colon\omega(\boldsymbol{z})\supset\widehat{I}\bigr\}=\bigl\{\boldsymbol{z}\colon\omega(\boldsymbol{z})\notin\widehat{\mathcal{K}}\bigr\}.\end{aligned}$$

The required identity follows by comparing the two formulae above. $\square$

Remark. The variety $X(\mathcal{K})$ is an example of an *arrangement of coordinate subspaces*, which will be studied further in Section **??**.

We finish this section by a result showing that the face ring determines its underlying simplicial complex:

Theorem 3.1.13 (Bruns–Gubeladze [**31**])**.** *Let $\mathbf{k}$ be a field, and $\mathcal{K}_1$ and $\mathcal{K}_2$ be two simplicial complexes on the vertex sets $[m_1]$ and $[m_2]$ respectively. Suppose $\mathbf{k}[\mathcal{K}_1]$ and $\mathbf{k}[\mathcal{K}_2]$ are isomorphic as $\mathbf{k}$-algebras. Then there exists a bijective map $[m_1] \to [m_2]$ which induces an isomorphism between $\mathcal{K}_1$ and $\mathcal{K}_2$.*

Proof. Let $f\colon \mathbf{k}[\mathcal{K}_1] \to \mathbf{k}[\mathcal{K}_2]$ be an isomorphism of $\mathbf{k}$-algebras. An easy argument shows that we can assume that f is a graded isomorphism (an exercise, or see [**31**, p. 316]).

Since f is graded, by restriction to the linear components we observe that $m_1 = m_2$ and that f is induced by a linear isomorphism $F\colon \mathbf{k}[m_1] \to \mathbf{k}[m_2]$. This is described by the commutative diagram

$$\begin{tikzcd}
\mathbf{k}[v_1,\ldots,v_{m_1}] \arrow[r, "F"] \arrow[d] & \mathbf{k}[v_1,\ldots,v_{m_2}] \arrow[d] \\
\mathbf{k}[\mathcal{K}_1] \arrow[r, "f"] & \mathbf{k}[\mathcal{K}_2]
\end{tikzcd}$$

By passing to the associated affine varieties, we observe that the isomorphism $f^*\colon X(\mathcal{K}_2) \to X(\mathcal{K}_1)$ is the restriction of the $\mathbf{k}$-linear isomorphism $F^*\colon \mathbf{k}^{m_2} \to \mathbf{k}^{m_1}$. This is described by the commutative diagram

$$\begin{tikzcd}
\mathbf{k}^{m_2} \arrow[r, "F^*"] & \mathbf{k}^{m_1} \\
X(\mathcal{K}_2) \arrow[r, "f^*"] \arrow[u] & X(\mathcal{K}_1) \arrow[u]
\end{tikzcd}$$

The isomorphism f^* establishes a bijective correspondence

$$\varPhi\colon \{\text{maximal faces of } \mathcal{K}_2\} \to \{\text{maximal faces of } \mathcal{K}_1\}$$

which is defined by the formula $f^*(S_I) = S_{\varPhi(I)}$, where I is a maximal face of $\mathcal{K}_2$. It is also clear that $|\varPhi(I)| = |I| = \dim S_I$.

We denote by $\mathcal{P}_1$ the intersection poset of the subspaces S_I, $I \in \mathcal{K}_1$, with respect to inclusion (i.e. the elements of $\mathcal{P}_1$ are nonempty intersections $S_{I_1} \cap \cdots \cap S_{I_k}$ with $I_j \in \mathcal{K}_1$). The poset $\mathcal{P}_1$ can be also viewed as the intersection poset of the maximal faces of $\mathcal{K}_1$. We define the poset $\mathcal{P}_2$ corresponding to $\mathcal{K}_2$ similarly. The correspondence $\varPhi$ obviously extends to an isomorphism of posets $\varPhi\colon \mathcal{P}_2 \to \mathcal{P}_1$, which preserves the dimension of spaces (or the number of elements in the intersections of maximal faces).

Now introduce the following equivalence relation on the vertex sets $[m_1]$ and $[m_2]$: for $i_1, i_2 \in [m_1]$ (or $j_1, j_2 \in [m_2]$) we put $i_1 \sim i_2$ if and only if the two sets of maximal faces $\mathcal{K}_1$ containing i_1 and i_2 respectively coincide (and similarly for j_1 and j_2). The equivalence classes in $[m_1]$ are the minimal (with respect to inclusion) nonempty intersections of maximal faces of $\mathcal{K}_1$, and similarly for $[m_2]$. Since $\varPhi$ is an isomorphism of posets, the two systems of equivalence classes are in natural bijective correspondence, and the corresponding equivalence classes have the same numbers of elements. This gives rise to the bijective map $\varphi\colon [m_2] \to [m_1]$ which satisfies the condition that $i \in I$ if and only if $\varphi(i) \in \varPhi(I)$, where $i \in [m_2]$ and $I \in \mathcal{K}_2$ is a maximal face. Since any face of a simplicial complex is contained in a maximal face, we finally obtain that $\psi = \varphi^{-1}\colon [m_1] \to [m_2]$ is the required map. $\square$

Exercises.

Exercise 3.1.14. Show that the Stanley–Reisner ideal $\mathcal{I}_\mathcal{K}$ is generated by quadratic monomials if and only if $\mathcal{K}$ is a flag complex.

Exercise 3.1.15 (see **[163**, (4.7)]**).** Let $\text{CAT}(\mathcal{K})$ be the face category of $\mathcal{K}$ (objects are simplices, morphisms are inclusions), $\text{CAT}^{op}(\mathcal{K})$ the opposite category (in which the morphisms are reverted), and CGA the category of commutative graded algebras. Consider the diagram

$$\begin{aligned}\mathbf{k}[\cdot]_\mathcal{K}\colon \text{CAT}^{op}(\mathcal{K}) &\longrightarrow \text{CGA},\\ I &\longmapsto \mathbf{k}[v_i\colon i\in I]\end{aligned}$$

whose value on a morphism $I\subset J$ is the surjection $\mathbf{k}[v_j\colon j\in J]\to\mathbf{k}[v_i\colon i\in I]$ sending each v_j with $j\notin I$ to zero. Show that

$$\mathbf{k}[\mathcal{K}]=\lim\mathbf{k}[\cdot]_\mathcal{K}$$

where the limit is taken in the category CGA. (See Appendix **??** for the details on categorical constructions.)

Exercise 3.1.16. If $\mathbf{k}[\mathcal{K}_1]$ and $\mathbf{k}[\mathcal{K}_2]$ are isomorphic as $\mathbf{k}$-algebras, then there is also a graded isomorphism $\mathbf{k}[\mathcal{K}_1]\to\mathbf{k}[\mathcal{K}_2]$. (Hint: show first that $\mathbf{k}[\mathcal{K}_1]$ and $\mathbf{k}[\mathcal{K}_2]$ are isomorphic as augmented $\mathbf{k}$-algebras, and then pass to the associated graded algebras with respect to the augmentation ideals.)

3.2. Tor-algebras and Betti numbers

The algebraic Betti numbers of the face ring $\mathbf{k}[\mathcal{K}]$ are the dimensions of the Tor-groups of $\mathbf{k}[\mathcal{K}]$ viewed as a module over the polynomial ring. These basic homological invariants of a simplicial complex $\mathcal{K}$ appear to be of great importance both for combinatorial commutative algebra and toric topology.

The face ring $\mathbf{k}[\mathcal{K}]$ acquires the canonical $\mathbf{k}[m]$-module structure via the quotient projection $\mathbf{k}[m]\to\mathbf{k}[\mathcal{K}]$. We therefore may consider the corresponding Tor-modules (see Appendix Section A.2):

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big)=\bigoplus_{i,j\geqslant 0}\operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big).$$

From Lemma A.2.9 we obtain that $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$ is a bigraded algebra in a natural way, and there is the following isomorphism of bigraded algebras:

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big)\cong H\big[\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[\mathcal{K}],d\big],$$

where the bigrading and differential on the right hand side are given by

$$\begin{aligned}\operatorname{bideg}u_i=(-1,2),\quad \operatorname{bideg}v_i=(0,2),\\ du_i=v_i,\quad dv_i=0.\end{aligned}\tag{3.1}$$

Definition 3.2.1. We refer to $\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$ as the Tor-*algebra* of a simplicial complex $\mathcal{K}$.

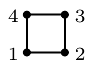
The *bigraded Betti numbers* of $\mathbf{k}[\mathcal{K}]$ are defined by

$$\beta^{-i,2j}\big(\mathbf{k}[\mathcal{K}]\big)=\dim_\mathbf{k}\operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big),\qquad\text{for } i,j\geqslant 0.\tag{3.2}$$

We also set

$$\beta^{-i}\big(\mathbf{k}[\mathcal{K}]\big)=\dim_\mathbf{k}\operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big)=\sum_j\beta^{-i,2j}\big(\mathbf{k}[\mathcal{K}]\big).$$

Consider the minimal resolution $[R_{\min}, d]$ of the $\mathbf{k}[m]$-module $\mathbf{k}[\mathcal{K}]$ (see Construction A.2.2). Then $R^0_{\min} \cong 1 \cdot \mathbf{k}[m]$ is a free module with one generator of degree 0. The basis of $R^{-1}_{\min}$ is a minimal generator set for $\mathcal{I}_{\mathcal{K}}$, and these minimal generators correspond to the missing faces of $\mathcal{K}$. Given a missing face $\{i_1, \dots, i_k\} \subset [m]$, denote by $r_{i_1,\dots,i_k}$ the corresponding generator of $R^{-1}_{\min}$. Then the map $d\colon R^{-1}_{\min} \to R^0_{\min}$ takes $r_{i_1,\dots,i_k}$ to $v_{i_1} \dots v_{i_k}$. By Proposition (A.2.5), $\beta^{-1,2j}(\mathbf{k}[\mathcal{K}])$ equals to the number of missing faces with j elements.

Example 3.2.2. Let $\mathcal{K} =$ 

, the boundary of a 4-gon. Then

$$\mathbf{k}[\mathcal{K}] \cong \mathbf{k}[v_1, \dots, v_4]/(v_1v_3, v_2v_4).$$

Let us construct a minimal resolution of $\mathbf{k}[\mathcal{K}]$. The module $R^0_{\min}$ has one generator 1 (of degree 0). The module $R^{-1}_{\min}$ has two generators r_{13} and r_{24} of degree 4, and the differential $d\colon R^{-1}_{\min} \to R^0_{\min}$ takes r_{13} to v_1v_3 and r_{24} to v_2v_4. The kernel $R^{-1}_{\min} \to R^0_{\min}$ is generated by one element $v_2v_4r_{13} - v_1v_3r_{24}$. Hence, $R^{-2}_{\min}$ has one generator of degree 8, which we denote by a, and the map $d\colon R^{-2}_{\min} \to R^{-1}_{\min}$ is injective and takes a to $v_2v_4r_{13} - v_1v_3r_{24}$. Thus, the minimal resolution is

$$0 \longrightarrow R^{-2}_{\min} \longrightarrow R^{-1}_{\min} \longrightarrow R^0_{\min} \longrightarrow M \longrightarrow 0,$$

where $\operatorname{rank} R^0_{\min} = \beta^{0,0}(\mathbf{k}[\mathcal{K}]) = 1$, $\operatorname{rank} R^{-1}_{\min} = \beta^{-1,4}(\mathbf{k}[\mathcal{K}]) = 2$, $\operatorname{rank} R^{-2}_{\min} = \beta^{-2,8}(\mathbf{k}[\mathcal{K}]) = 1$.

The following fundamental result of Hochster reduces the calculation of the Betti numbers $\beta^{-i,2j}(\mathbf{k}[\mathcal{K}])$ to the calculation of reduced simplicial cohomology of full subcomplexes in $\mathcal{K}$.

Theorem 3.2.3 (Hochster [**111**])**.** *We have*

$$\operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}[\mathcal{K}], \mathbf{k}\bigr) = \bigoplus_{J \subset [m]\colon |J|=j} \widetilde{H}^{j-i-1}(\mathcal{K}_J; \mathbf{k}),$$

where $\mathcal{K}_J$ is the full subcomplex of $\mathcal{K}$ obtained by restricting to $J \subset [m]$. We assume $\widetilde{H}^{-1}(\mathcal{K}_\varnothing; \mathbf{k}) = \mathbf{k}$ above.

We shall give a proof of Hochster's formula following [**159**]. The idea is first to reduce the Koszul algebra $[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}], d]$ to its certain finite dimensional quotient $R^*(\mathcal{K})$, without changing the cohomology, and then identify $R^*(\mathcal{K})$ with the sum of simplicial cochain complexes of all full subcomplexes in $\mathcal{K}$. The algebra $R^*(\mathcal{K})$ will be also used in the cohomology calculations for moment-angle complexes in Chapter **??**.

We use simplified notation u_Jv_I for a monomial $u_J \otimes v_I$ in the Koszul algebra $\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}]$.

Construction 3.2.4. We introduce the quotient algebra

$$R^*(\mathcal{K}) = \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}]\big/(v_i^2 = u_iv_i = 0,\ 1 \leqslant i \leqslant m),$$

with differential and bigrading defined by (3.1), together with the quotient projection

$$\varrho\colon \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}] \to R^*(\mathcal{K}).$$

By definition, the algebra $R^*(\mathcal{K})$ has a $\mathbf{k}$-vector space basis consisting of monomials u_Jv_I where $J \subset [m]$, $I \in \mathcal{K}$ and $J \cap I = \varnothing$. Hence, we have a $\mathbf{k}$-linear map

$$\iota\colon R^*(\mathcal{K}) \to \Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}],$$

which commutes with the differentials, and therefore defines a homomorphism of bigraded differential **k**-vector spaces satisfying the relation $\varrho \cdot \iota = \mathrm{id}$. Note that ι is not a map of algebras.

Lemma 3.2.5. *The projection homomorphism* $\varrho\colon \Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[\mathcal{K}] \to R^*(\mathcal{K})$ *induces an isomorphism in cohomology.*

PROOF. The argument is similar to that used for the Koszul resolution (see Construction A.2.3). We shall construct a cochain homotopy between the maps id and $\iota\cdot\varrho$ from $\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[\mathcal{K}]$ to itself, that is, a map s satisfying the identity

$$ds + sd = \mathrm{id} - \iota\cdot\varrho. \tag{3.3}$$

We first consider the case $\mathcal{K} = \Delta^{m-1}$. Then $\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[\Delta^{m-1}]$ is the Koszul resolution (A.5), which will be denoted by

$$E = E_m = \Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[v_1,\dots,v_m], \tag{3.4}$$

and the algebra $R^*(\Delta^{m-1})$ is isomorphic to

$$\bigl(\Lambda[u]\otimes\mathbf{k}[v]/(v^2,uv)\bigr)^{\otimes m}. \tag{3.5}$$

For $m=1$, we define the map $s_1\colon E_1^{0,*} = \mathbf{k}[v] \to E_1^{-1,*}$ by the formula

$$s_1(a_0 + a_1v + \dots + a_jv^j) = (a_2v + a_3v^2 + \dots + a_jv^{j-1})u.$$

It gives the required cochain homotopy. Indeed, any element of E_1 is either x or xu with $x = a_0+a_1v+\dots+a_jv^j \in E^{0,*}$. In the first case we have $ds_1x = x-a_0-a_1v = x-\iota\varrho x$, and $s_1dx = 0$. In the second case, i.e. for $xu\in E_1^{-1,*}$, we have $ds_1(xu) = 0$, and $s_1d(xu) = xu - a_0u = xu - \iota\varrho(xu)$. In either case (3.3) holds.

Now we may assume by induction that a cochain homotopy $s_m\colon E_m\to E_m$ has been already constructed for $m = k-1$. Since $E_k = E_{k-1}\otimes E_1$, $\varrho_k = \varrho_{k-1}\otimes\varrho_1$ and $\iota_k = \iota_{k-1}\otimes\iota_1$, a direct calculation shows that the map

$$s_k = s_{k-1}\otimes\mathrm{id} + \iota_{k-1}\varrho_{k-1}\otimes s_1$$

is a cochain homotopy between id and $\iota_k\varrho_k$, which finishes the proof for $\mathcal{K} = \Delta^{m-1}$.

In the case of arbitrary $\mathcal{K}$ the algebras $\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[\mathcal{K}]$ and $R^*(\mathcal{K})$ are obtained by factorising (3.4) and (3.5) respectively by the ideal $\mathcal{I}_\mathcal{K}$. Then we have

$$\iota\varrho(\mathcal{I}_\mathcal{K})\subset\mathcal{I}_\mathcal{K},\quad d(\mathcal{I}_\mathcal{K})\subset\mathcal{I}_\mathcal{K},\quad s(\mathcal{I}_\mathcal{K})\subset\mathcal{I}_\mathcal{K}$$

(note that the last inclusion does not hold for the map s used in the proof of the acyclicity of the Koszul resolution). Thus, identity (3.3) still holds. □

As an immediate consequence of Lemma 3.2.5 we obtain

Corollary 3.2.6. *We have that* $\beta^{-i,2j}\bigl(\mathbf{k}[\mathcal{K}]\bigr) = 0$ *if* $i > m$ *or* $j > m$.

PROOF. Indeed, we have $R^{-i,2j}(\mathcal{K}) = 0$ if either i or j is greater than m. □

Now, in order to prove Theorem 3.2.3, we need to show that cohomology of $R^*(\mathcal{K})$ is isomorphic to the direct sum of reduced cohomology of full subcomplexes on the right hand side of Hochster's formula. We shall see that this is true even without passing to cohomology, i.e. $R^*(\mathcal{K})$ is isomorphic to $\bigoplus_{I\subset m} C^*(\mathcal{K}_I)$, with the appropriate shift in dimensions, where C^* denotes the simplicial cochain groups. To do this, it is convenient to refine the grading in $\mathbf{k}[\mathcal{K}]$ as follows.

Construction 3.2.7 (multigraded structure in face rings and Tor-algebras)**.** A *multigrading* (more precisely, an $\mathbb{N}^m$-grading) is defined in $\mathbf{k}[v_1,\ldots,v_m]$ by setting

$$\operatorname{mdeg} v_1^{i_1}\cdots v_m^{i_m} = (2i_1,\ldots,2i_m).$$

Since $\mathbf{k}[\mathcal{K}]$ is the quotient of the polynomial ring by a monomial ideal, it inherits the multigrading. We may assume that all free modules in resolution (A.2) are multigraded and the differentials preserve the multidegree. Then the algebra $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$ acquires the canonical $\mathbb{Z}\oplus\mathbb{N}^m$-grading, i.e.

$$\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[\mathcal{K}],\mathbf{k}\bigr) = \bigoplus_{i\geqslant 0,\ \boldsymbol{a}\in\mathbb{N}^m}\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[\mathcal{K}],\mathbf{k}\bigr).$$

The differential algebra $R^*(\mathcal{K})$ also acquires a $\mathbb{Z}\oplus\mathbb{N}^m$-grading, and Lemma 3.2.5 implies that

$$\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[\mathcal{K}],\mathbf{k}\bigr)\cong H^{-i,2\boldsymbol{a}}[R^*(\mathcal{K}),d]. \tag{3.6}$$

We may view a subset $J\subset[m]$ as a $(0,1)$-vector in $\mathbb{N}^m$ whose jth coordinate is 1 if $j\in J$ and is 0 otherwise. Then there is the following multigraded version of Hochster's formula:

Theorem 3.2.8. *For any subset $J\subset[m]$ we have*

$$\operatorname{Tor}^{-i,2J}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[\mathcal{K}],\mathbf{k}\bigr)\cong \widetilde{H}^{|J|-i-1}(\mathcal{K}_J),$$

and $\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})=0$ *if $\boldsymbol{a}$ is not a $(0,1)$-vector.*

Proof of Theorem 3.2.3 and Theorem 3.2.8. Let $C^q(\mathcal{K}_J)$ denote the qth simplicial cochain group with coefficients in $\mathbf{k}$. Denote by $\alpha_L\in C^{p-1}(\mathcal{K}_J)$ the basis cochain corresponding to an oriented simplex $L=(l_1,\ldots,l_p)\in\mathcal{K}_J$; it takes value 1 on L and vanishes on all other simplices. Now we define a $\mathbf{k}$-linear map

$$\begin{aligned} f\colon C^{p-1}(\mathcal{K}_J) &\longrightarrow R^{p-|J|,2J}(\mathcal{K}),\\ \alpha_L &\longmapsto \varepsilon(L,J)\,u_{J\setminus L}v_L, \end{aligned} \tag{3.7}$$

where $\varepsilon(L,J)$ is the sign defined by

$$\varepsilon(L,J)=\prod_{j\in L}\varepsilon(j,J),$$

and $\varepsilon(j,J)=(-1)^{r-1}$ if j is the rth element of the set $J\subset[m]$, written in increasing order. Obviously, f is an isomorphism of $\mathbf{k}$-vector spaces, and a direct check shows that it commutes with the differentials. Indeed, we have

$$\begin{aligned} f(d\alpha_L) &= f\Bigl(\sum_{j\in J\setminus L,\ j\cup L\in\mathcal{K}_J}\varepsilon(j,j\cup L)\,\alpha_{j\cup L}\Bigr)\\ &=\sum_{j\in J\setminus L}\varepsilon(j\cup L,J)\varepsilon(j,j\cup L)\,u_{J\setminus(j\cup L)}v_{j\cup L} \end{aligned}$$

(note that $v_{j\cup L}\in\mathbf{k}[\mathcal{K}]$, and hence it is zero unless $j\cup L\in\mathcal{K}_J$). On the other hand,

$$df(\alpha_L)=\sum_{j\in J\setminus L}\varepsilon(L,J)\varepsilon(j,J\setminus L)u_{J\setminus(j\cup L)}v_{j\cup L}.$$

By the definition of $\varepsilon(L,J)$,

$$\varepsilon(j\cup L,J)\varepsilon(j,j\cup L)=\varepsilon(L,J)\varepsilon(j,J)\varepsilon(j,j\cup L)=\varepsilon(L,J)\varepsilon(j,J\setminus L),$$

which implies that $f(d\alpha_L) = df(\alpha_L)$. Therefore, f together with the map $\mathbf{k} \to R^{-|J|,2J}(\mathcal{K})$, $1 \mapsto u_J$, defines an isomorphism of cochain complexes

$$\begin{array}{ccccccccccc}
0 \to & \mathbf{k} & \xrightarrow{d} & C^0(\mathcal{K}_J) & \xrightarrow{d} & \cdots & \xrightarrow{d} & C^{p-1}(\mathcal{K}_J) & \xrightarrow{d} \cdots \\
& \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & \\
0 \to & R^{-|J|,2J}(\mathcal{K}) & \xrightarrow{d} & R^{1-|J|,2J}(\mathcal{K}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & R^{p-|J|,2J}(\mathcal{K}) & \xrightarrow{d} \cdots
\end{array}$$

Then it follows from (3.6) that

$$\widetilde{H}^{p-1}(K_J) \cong \operatorname{Tor}^{p-|J|,2J}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}[\mathcal{K}],\mathbf{k}\bigr),$$

which is equivalent to the first isomorphism of Theorem 3.2.8. Since $R^{-i,2\boldsymbol{a}} = 0$ if $\boldsymbol{a}$ is not a $(0,1)$-vector, $\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$ vanishes for such $\boldsymbol{a}$. $\square$

Since $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}],\mathbf{k})$ is an algebra, the isomorphisms of Theorem 3.2.3 turn the direct sum

$$\bigoplus_{\substack{p\geqslant 0\\ J\subset[m]}} \widetilde{H}^{p-1}(\mathcal{K}_J) \tag{3.8}$$

into a (multigraded) $\mathbf{k}$-algebra. Consider the product in the simplicial cochains of full subcomplexes given by

$$\begin{aligned}
\mu\colon C^{p-1}(\mathcal{K}_I)\otimes C^{q-1}(\mathcal{K}_J) &\longrightarrow C^{p+q-1}(\mathcal{K}_{I\cup J}),\\
\alpha_L\otimes\alpha_M &\longmapsto \begin{cases} \alpha_{L\sqcup M}, & \text{if } I\cap J=\varnothing;\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}\tag{3.9}$$

Here $\alpha_{L\sqcup M} \in C^{p+q-1}(\mathcal{K}_{I\sqcup J})$ denotes the basis simplicial cochain corresponding to $L \sqcup M$ if the latter is a simplex of $\mathcal{K}_{I\sqcup J}$ and zero otherwise. If $I \cap J = \varnothing$, then $\mathcal{K}_{I\sqcup J}$ is a subcomplex in the join $\mathcal{K}_I * \mathcal{K}_J$, and the above product is the restriction to $\mathcal{K}_{I\sqcup J}$ of the standard exterior product

$$C^{p-1}(\mathcal{K}_I)\otimes C^{q-1}(\mathcal{K}_J) \longrightarrow C^{p+q-1}(\mathcal{K}_I * \mathcal{K}_J).$$

Proposition 3.2.9. *The product in the direct sum* (3.8) *induced by the isomorphisms from Hochster's theorem coincides up to a sign with the product given by* (3.9).

Proof. This is a direct calculation. We use the isomorphism f given by (3.7):

$$\alpha_L\cdot\alpha_M = f^{-1}\bigl(f(\alpha_L)\cdot f(\alpha_M)\bigr) = f^{-1}\bigl(\varepsilon(L,I)\,u_{I\setminus L}v_L\,\varepsilon(M,J)\,u_{J\setminus M}v_M\bigr)$$

If $I\cap J\neq\varnothing$, then the product $u_{I\setminus L}v_Lu_{J\setminus M}v_M$ is zero in $R^*(\mathcal{K})$. Otherwise we have that $u_{I\setminus L}v_Lu_{J\setminus M}v_M = \zeta\,u_{(I\cup J)\setminus(L\cup M)}v_{L\cup M}$, where $\zeta = \prod_{k\in I\setminus L}\varepsilon(k, k\cup J\setminus M)$, and we can continue the above identity as

$$\alpha_L\cdot\alpha_M = \varepsilon(L,I)\,\varepsilon(M,J)\,\zeta\,\varepsilon(L\cup M, I\cup J)\,\alpha_{L\sqcup M}.$$

Note that this calculation also gives the explicit value for the correcting sign, but we shall not need this. $\square$

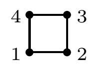
Let P be a simple polytope. The multigraded components of $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k}[P],\mathbf{k})$ can be expressed directly in terms of P as follows. Let $\{F_1,\dots,F_m\}$ be the set of facets of P. Given $I\subset[m]$, we define the following subset of the boundary of P:

$$P_I = \bigcup_{i\in I} F_i.$$

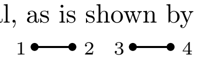
Proposition 3.2.10. *For any subset $J \subset [m]$ we have*
$$\operatorname{Tor}^{-i,2J}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[P],\mathbf{k}\bigr) \cong \widetilde{H}^{|J|-i-1}(P_J),$$
and $\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbf{k}[m]}(\mathbf{k}[P],\mathbf{k}) = 0$ *if $\boldsymbol{a}$ is not a $(0,1)$-vector.*

Proof. Let $\mathcal{K} = \mathcal{K}_P$ be the nerve complex of P. Then the statement follows from Theorem 3.2.8 and the fact that $\mathcal{K}_J$ is a deformation retract of P_J. The latter is because P is simple, and therefore, $P_J = \bigcup_{i\in J} F_J = \bigcup_{i\in J} \operatorname{st}_{\mathcal{K}'}\{i\}$ (by Proposition **??**), which is the combinatorial neighbourhood of $(\mathcal{K}_J)'$ in $\mathcal{K}'$. □

For a description of the multiplication in $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k}[P],\mathbf{k})$ in terms of P, see Exercise 3.2.13.

Example 3.2.11. 1. Let P be a 4-gon, so that $\mathcal{K}_P =$ 

, as in Example 3.2.2. This time we calculate the Betti numbers $\beta^{-i,2j}(\mathbf{k}[P])$ using Hochster's formula. We have that
$$\begin{aligned}
\beta^{0,0}\bigl(\mathbf{k}[P]\bigr) &= \dim \widetilde{H}^0(\varnothing) = 1 && 1,\\
\beta^{-1,4}\bigl(\mathbf{k}[P]\bigr) &= \dim \widetilde{H}^0\bigl(P_{\{1,3\}}\bigr) \oplus \widetilde{H}^0\bigl(P_{\{2,4\}}\bigr) = 2 && u_1v_3,\, u_2v_4\\
\beta^{-2,8}\bigl(\mathbf{k}[P]\bigr) &= \dim \widetilde{H}^1\bigl(P_{\{1,2,3,4\}}\bigr) = 1 && u_1u_2v_3v_4,
\end{aligned}$$
where in the right column we include cocycles in the Koszul algebra $\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[P]$ representing generators of the corresponding cohomology groups. All other Betti numbers are zero. We have a nontrivial product $[u_1v_3]\cdot[u_2v_4] = [u_1u_2v_3v_4]$; all other products of positive-dimensional classes are zero. Note that in this example all Tor-groups have bases represented by monomials in the Koszul algebra. This is not the case in general, as is shown by the next example.

2. Now let $\mathcal{K} =$ 

be the union of two segments. Then the generator of
$$\operatorname{Tor}^{-3,8}_{\mathbf{k}[v_1,\ldots,v_4]}\bigl(\mathbf{k}[\mathcal{K}],\mathbf{k}\bigr) \cong \widetilde{H}^0\bigl(\mathcal{K}_{\{1,2,3,4\}}\bigr) \cong \mathbf{k}$$
is represented by the cocycle $u_1u_2u_3v_4 - u_1u_2u_4v_3$ in the Koszul algebra, and it cannot be represented by a monomial.

3. Let us calculate the Betti numbers (both bigraded and multigraded) of $\mathbf{k}[\mathcal{K}]$ for the complex shown in Fig. 3.1, using Hochster's formula. We have
$$\begin{aligned}
\beta^{0,0} &= \dim \widetilde{H}^0(\varnothing) = 1,\\
\beta^{-1,4} &= \beta^{-1,(2,0,0,0,2)} + \beta^{-1,(0,0,2,2,0)} = \dim \widetilde{H}^0\bigl(\mathcal{K}_{\{1,5\}}\bigr) \oplus \widetilde{H}^0\bigl(\mathcal{K}_{\{3,4\}}\bigr) = 2,\\
\beta^{-1,6} &= \beta^{-1,(2,2,2,0,0)} + \beta^{-1,(0,2,0,2,2)} = \dim \widetilde{H}^1\bigl(\mathcal{K}_{\{1,2,3\}}\bigr) \oplus \widetilde{H}^1\bigl(\mathcal{K}_{\{2,4,5\}}\bigr) = 2,\\
\beta^{-2,8} &= \beta^{-2,(0,2,2,2,2)} + \beta^{-2,(2,0,2,2,2)} + \cdots + \beta^{-2,(2,2,2,2,0)}\\
&\qquad\qquad = \dim \widetilde{H}^1\bigl(\mathcal{K}_{\{2,3,4,5\}}\bigr) \oplus \cdots \oplus \widetilde{H}^1\bigl(\mathcal{K}_{\{1,2,3,4\}}\bigr) = 5,\\
\beta^{-3,10} &= \beta^{-3,(2,2,2,2,2)} = \dim \widetilde{H}^1\bigl(\mathcal{K}_{\{1,2,3,4,5\}}\bigr) = 2.
\end{aligned}$$
All other Betti numbers are zero.

4. Let $\mathcal{K}$ be a triangulation of the real projective plane $\mathbb{R}P^2$ with m vertices (the minimal example has $m = 6$, see Fig. 3.2, where the vertices with the same labels are identified, and the boundary edges are identified according to the orientation shown). Then, by Hochster's formula,
$$\operatorname{Tor}^{3-m,2m}_{\mathbf{k}[v_1,\ldots,v_m]}\bigl(\mathbf{k}[\mathcal{K}],\mathbf{k}\bigr) = \widetilde{H}^2(\mathcal{K}_{[m]};\mathbf{k}) = \widetilde{H}^2(\mathbb{R}P^2;\mathbf{k}) = 0$$

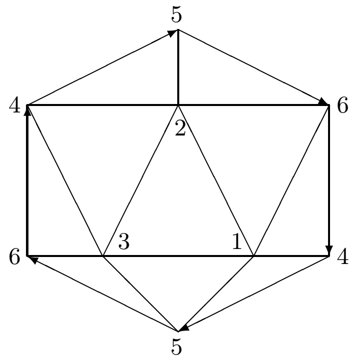


FIGURE 3.2. 6-vertex triangulation of $\mathbb{R}P^2$.

if the characteristic of $\mathbf{k}$ is not 2. On the other hand,

$$\operatorname{Tor}^{3-m,2m}_{\mathbb{Z}_2[v_1,\ldots,v_m]}\bigl(\mathbb{Z}_2[\mathcal{K}],\mathbb{Z}_2\bigr) = \widetilde{H}^2(\mathcal{K}_{[m]};\mathbb{Z}_2) = \widetilde{H}^2(\mathbb{R}P^2;\mathbb{Z}_2) = \mathbb{Z}_2.$$

This example shows that the Tor-groups of $\mathbf{k}[\mathcal{K}]$, and even the algebraic Betti numbers, depend on $\mathbf{k}$. A similar example shows that $\operatorname{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}],\mathbb{Z})$ may have an arbitrary amount of additive torsion. (This is a well-known fact for the usual cohomology of spaces, and so we may take $\mathcal{K}$ to be a triangulation of a space with the appropriate torsion in cohomology.)

Exercises.

Exercise 3.2.12. Let P be a pentagon. Calculate the bigraded Betti numbers of $\mathbf{k}[P]$ and the multiplication in $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k}[P],\mathbf{k})$

(a) using algebra $R^*(\mathcal{K}_P)$ and Lemma 3.2.5;
(b) using Hochster's theorem and Proposition 3.2.9,

and compare the results.

Exercise 3.2.13. Use Proposition 3.2.10 and the isomorphism

$$\widetilde{H}^{|J|-i-1}(P_J) \cong H^{|J|-i}(P,P_J)$$

to show that the multiplication induced from $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k}[P],\mathbf{k})$ in the direct sum

$$\bigoplus_{\substack{p\geqslant 0\\ J\subset[m]}} H^p(P,P_J)$$

comes from the standard exterior multiplication

$$H^p(P,P_I)\otimes H^q(P,P_J) \longrightarrow H^{p+q}(P,P_I\cup P_J)$$

when $I\cap J=\varnothing$ and is zero otherwise.

Exercise 3.2.14. Complete the details in the following algebraic proof of the Alexander duality (Theorem **??**); this argument ascends to the original work of Hochster [**111**]:

1. Choose $J\notin\mathcal{K}$, that is, $\widehat{J}=[m]\setminus J\in\widehat{\mathcal{K}}$, and show that for any $L=\{l_1,\ldots,l_q\}\subset J$,

$$J\setminus L\notin\mathcal{K} \iff L\in\operatorname{lk}_{\widehat{\mathcal{K}}}\widehat{J}.$$

2. Consider the Koszul algebra

$$S(\mathcal{K}) = \bigl[\Lambda[u_1, \ldots, u_m] \otimes \mathcal{I}_{\mathcal{K}}, d\bigr]$$

of the Stanley–Reisner ideal $\mathcal{I}_{\mathcal{K}}$ (see Lemma A.2.9 and the remark after it), and show that its multigraded component $S^{-q,2J}(\mathcal{K})$ has a $\mathbf{k}$-basis consisting of monomials $u_L v_{J\setminus L}$ where $L \in \operatorname{lk}_{\widehat{\mathcal{K}}} \widehat{J}$.

3. Consider the $\mathbf{k}$-vector space isomorphism

$$\begin{aligned} g\colon C_{q-1}(\operatorname{lk}_{\widehat{\mathcal{K}}} \widehat{J}) &\longrightarrow S^{-q,2J}(\mathcal{K}),\\ [L] &\longmapsto u_L v_{J\setminus L}, \end{aligned}$$

where $[L] \in C_{q-1}(\operatorname{lk}_{\widehat{\mathcal{K}}} \widehat{J})$ is the basis simplicial chain corresponding to L. Show that g commutes with the differentials, and therefore defines an isomorphism of chain complexes (in analogy with (3.7), but with no correction sign).

4. Deduce that

$$\widetilde{H}_{q-1}(\operatorname{lk}_{\widehat{\mathcal{K}}} \widehat{J}) \cong \operatorname{Tor}^{-q,2J}_{\mathbf{k}[m]}(\mathcal{I}_{\mathcal{K}}, \mathbf{k}) \cong \operatorname{Tor}^{-q-1,2J}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \widetilde{H}^{|J|-q-2}(\mathcal{K}_J),$$

where the first isomorphism is obtained by passing to homology in step 3, the second follows from the long exact sequence of Proposition A.2.4 (e), and the third is Theorem 3.2.3. It remains to note that the resulting isomorphism is equivalent to that of Corollary **??**.

3.3. Cohen–Macaulay complexes

It is usually quite difficult to determine whether a given ring is Cohen–Macaulay (see Appendix Section A.3). One of the key results of combinatorial commutative algebra, the *Reisner Theorem*, gives an effective criterion for the Cohen–Macaulayness of face rings, in terms of simplicial cohomology of $\mathcal{K}$. A reformulation of Reisner's criterion, due to Munkres and Stanley, tells us that the Cohen–Macaulayness of $\mathbf{k}[\mathcal{K}]$ is a topological property of $\mathcal{K}$, i.e. it depends only on the topology of the realisation $|\mathcal{K}|$. These results have many important applications in both combinatorial commutative algebra and toric topology.

Here we assume that $\mathbf{k}$ is a field, unless otherwise stated. Note that if $\mathcal{K}$ is of dimension $n-1$, then the Krull dimension of $\mathbf{k}[\mathcal{K}]$ is n (an exercise). We start by the following combinatorial description of homogeneous systems of parameters (hsop's) in $\mathbf{k}[\mathcal{K}]$ in terms of the restriction homomorphisms $s_I\colon \mathbf{k}[\mathcal{K}] \to \mathbf{k}[v_i\colon i \in I]$ (see Proposition 3.1.8).

Lemma 3.3.1. *Let $\mathcal{K}$ be a simplicial complex of dimension $n-1$. A sequence of homogeneous elements $\boldsymbol{t} = (t_1, \ldots, t_n)$ of $\mathbf{k}[\mathcal{K}]$ is a homogeneous system of parameters if and only if*

$$\dim_{\mathbf{k}}\bigl(\mathbf{k}[v_i\colon i \in I]/s_I(\boldsymbol{t})\bigr) < \infty$$

for each simplex $I \in \mathcal{K}$, where $s_I(\boldsymbol{t})$ is the image of the sequence $\boldsymbol{t}$ under the restriction map s_I.

Proof. Assume that $\boldsymbol{t}$ is an hsop. By applying the right exact functor $\otimes_{\mathbf{k}[\boldsymbol{t}]}$ to the epimorphism $s_I\colon \mathbf{k}[\mathcal{K}] \to \mathbf{k}[v_i\colon i \in I]$ we obtain that $\mathbf{k}[\mathcal{K}]/\boldsymbol{t} \to \mathbf{k}[v_i\colon i \in I]/s_I(\boldsymbol{t})$ is also an epimorphism. Hence,

$$\dim_{\mathbf{k}}\bigl(\mathbf{k}[v_i\colon i \in I]/s_I(\boldsymbol{t})\bigr) \leqslant \dim_{\mathbf{k}}\bigl(\mathbf{k}[\mathcal{K}]/\boldsymbol{t}\bigr) < \infty.$$

For the opposite statement, assume that

$$\dim_{\mathbf{k}} \bigoplus_{I\in\mathcal{K}} \mathbf{k}[v_i\colon i\in I]/s_I(\boldsymbol{t}) < \infty.$$

Consider the short exact sequence of $\mathbf{k}[\boldsymbol{t}]$-modules

$$0 \longrightarrow \mathbf{k}[\mathcal{K}] \longrightarrow \bigoplus_{I\in\mathcal{K}} \mathbf{k}[v_i\colon i\in I] \longrightarrow Q \longrightarrow 0,$$

where Q is the quotient module, and the following fragment of the corresponding long exact sequence for Tor (Proposition A.2.4 (e)):

$$\cdots \longrightarrow \operatorname{Tor}^{-1}_{\mathbf{k}[\boldsymbol{t}]}(Q,\mathbf{k}) \longrightarrow \mathbf{k}[\mathcal{K}]/\boldsymbol{t} \longrightarrow \bigoplus_{I\in\mathcal{K}} \mathbf{k}[v_i\colon i\in I]/s_I(\boldsymbol{t}) \longrightarrow \cdots .$$

Since $\bigoplus_{I\in\mathcal{K}} \mathbf{k}[v_i\colon i\in I]$ is a finitely generated $\mathbf{k}[\boldsymbol{t}]$-module, its quotient Q is also finitely generated. Hence, $\dim_{\mathbf{k}} \operatorname{Tor}^{-1}_{\mathbf{k}[\boldsymbol{t}]}(Q,\mathbf{k}) < \infty$ (see Proposition A.2.5). Then it follows from the exact sequence above that $\dim_{\mathbf{k}}\bigl(\mathbf{k}[\mathcal{K}]/\boldsymbol{t}\bigr) < \infty$. Therefore, $\boldsymbol{t}$ is an hsop in $\mathbf{k}[\mathcal{K}]$. □

Recall that we refer to $\boldsymbol{t} = (t_1,\ldots,t_n)$ as linear if $\deg t_i = 2$.

Corollary 3.3.2. *A linear sequence $\boldsymbol{t} = (t_1,\ldots,t_n) \subset \mathbf{k}[\mathcal{K}]$, $\dim\mathcal{K} = n-1$, is an lsop if and only if its restriction $s_I(\boldsymbol{t})$ to any simplex of $\mathbf{k}[\mathcal{K}]$ generates the polynomial ring $\mathbf{k}[v_i\colon i\in I]$.*

Proof. Indeed, if $\boldsymbol{t}$ is linear, then the condition $\dim_{\mathbf{k}} \mathbf{k}[v_i\colon i\in I]/s_I(\boldsymbol{t}) < \infty$ is equivalent to that $\mathbf{k}[v_i\colon i\in I]/s_I(\boldsymbol{t}) \cong \mathbf{k}$. □

Note that it is enough to verify the condition of Lemma 3.3.1 for maximal simplices $I\in\mathcal{K}$ only.

Definition 3.3.3. $\mathcal{K}$ is a *Cohen–Macaulay complex over a field* $\mathbf{k}$ if $\mathbf{k}[\mathcal{K}]$ is a Cohen–Macaulay algebra. We say that $\mathcal{K}$ is a *Cohen–Macaulay complex over* $\mathbb{Z}$, or shortly a *Cohen–Macaulay complex*, if $\mathbf{k}[\mathcal{K}]$ is a Cohen–Macaulay algebra for $\mathbf{k}=\mathbb{Q}$ and any finite field.

Remark. We shall often consider $\mathbf{k}[\mathcal{K}]$ as a $\mathbf{k}[m]$-module rather than a $\mathbf{k}$-algebra. However, this does not affect regular sequences and the Cohen–Macaulay property: it is an easy exercise that a sequence $\boldsymbol{t}\subset\mathbf{k}[m]$ is $\mathbf{k}[m]$-regular for $\mathbf{k}[\mathcal{K}]$ as a $\mathbf{k}[m]$-module if and only the image of $\boldsymbol{t}$ in $\mathbf{k}[\mathcal{K}]$ is $\mathbf{k}[\mathcal{K}]$-regular. In particular, $\mathbf{k}[\mathcal{K}]$ is a Cohen–Macaulay algebra if and only if it is a Cohen–Macaulay $\mathbf{k}[m]$-module. We shall therefore not distinguish between these two notions.

Example 3.3.4. Let $\mathcal{K} = \partial\Delta^2$. Then $\mathbf{k}[\mathcal{K}] = \mathbf{k}[v_1,v_2,v_3]/(v_1v_2v_3)$ and the Krull dimension is $\dim\mathbf{k}[\mathcal{K}] = 2$. The elements $v_1,v_2\in\mathbf{k}[\mathcal{K}]$ are algebraically independent, but do not form an lsop, since $\mathbf{k}[\mathcal{K}]/(v_1,v_2)\cong\mathbf{k}[v_3]$ and $\dim\bigl(\mathbf{k}[\mathcal{K}]/(v_1,v_2)\bigr) = 1$. On the other hand, the elements $t_1 = v_1-v_3$, $t_2 = v_2-v_3$ form an lsop, since $\mathbf{k}[\mathcal{K}]/(t_1,t_2)\cong\mathbf{k}[t]/t^3$. It is easy to see that $\mathbf{k}[\mathcal{K}]$ is a free $\mathbf{k}[t_1,t_2]$-module on the basis $\{1,v_1,v_1^2\}$. Therefore, $\mathbf{k}[\mathcal{K}]$ is a Cohen–Macaulay ring and (t_1,t_2) is a regular sequence.

There is the following homological characterisation of Cohen–Macaulay complexes:

Proposition 3.3.5. *The following conditions are equivalent for a simplicial complex $\mathcal{K}$ of dimension $n-1$ with m vertices:*

(a) *$\mathcal{K}$ is Cohen–Macaulay over a field $\mathbf{k}$;*
(b) *$\beta^{-i}(\mathbf{k}[\mathcal{K}])=0$ for $i>m-n$ and $\beta^{-(m-n)}(\mathbf{k}[\mathcal{K}])\neq 0$.*

PROOF. By definition, condition (a) is that $\operatorname{depth}\mathbf{k}[\mathcal{K}]=n$. Condition (b) is equivalent to that $\operatorname{pdim}\mathbf{k}[\mathcal{K}]=m-n$, by Proposition A.2.6. Since $\operatorname{depth}\mathbf{k}[m]=m$, the two conditions are equivalent by Theorem A.3.6. □

Example 3.3.6. Let $\mathcal{K}$ be the 6-vertex triangulation of $\mathbb{R}P^2$, see Example 3.2.11.4 and Figure 3.2. Then $m-n=3$, and, by Theorem 3.2.3,

$$\beta^{-4}\bigl(\mathbb{Z}_2[\mathcal{K}]\bigr)=\dim_{\mathbb{Z}_2}\widetilde{H}^1(\mathbb{R}P^2;\mathbb{Z}_2)=1,$$

so $\mathcal{K}$ is not Cohen–Macaulay over $\mathbb{Z}_2$. On the other hand, a similar calculation shows that if the characteristic of $\mathbf{k}$ is not 2, then $\beta^{-i}(\mathbf{k}[\mathcal{K}])=0$ for $i>3$ and $\beta^{-3}(\mathbf{k}[\mathcal{K}])=6$, i.e. $\mathcal{K}$ is Cohen–Macaulay over such fields.

Proposition 3.3.7 (Stanley)**.** *If $\mathcal{K}$ is a Cohen–Macaulay complex of dimension $n-1$, then $\boldsymbol{h}(\mathcal{K})=(h_0,\dots,h_n)$ is an M-vector (see Definition **??**).*

PROOF. Let $\mathcal{K}$ be Cohen–Macaulay, and let $\boldsymbol{t}=(t_1,\dots,t_n)$ be an lsop in $\mathbf{k}[\mathcal{K}]$, where $\mathbf{k}$ is a field of zero characteristic. Then, by Proposition A.3.12,

$$F\bigl(\mathbf{k}[\mathcal{K}],\lambda\bigr)=\frac{F\bigl(\mathbf{k}[\mathcal{K}]/\boldsymbol{t};\lambda\bigr)}{(1-\lambda^2)^n}.$$

On the other hand, the Poincaré series of $\mathbf{k}[\mathcal{K}]$ is given by Theorem 3.1.10, which implies that

$$F\bigl(\mathbf{k}[\mathcal{K}]/\boldsymbol{t};\lambda\bigr)=h_0+h_1\lambda^2+\dots+h_n\lambda^{2n}.$$

Now, $A=\mathbf{k}[\mathcal{K}]/\boldsymbol{t}$ is a graded algebra generated by its degree-two elements and $\dim_{\mathbf{k}}A^{2i}=h_i$, so $(h_0,\dots,h_n)$ is an M-vector by definition. □

Remark. According to a result of Stanley [**184**, Theorem II.3.3], if $(h_0,\dots,h_n)$ is an M-vector, then there exists an $(n-1)$-dimensional Cohen–Macaulay complex $\mathcal{K}$ such that $h_i(\mathcal{K})=h_i$. Together with Proposition 3.3.7, this gives a complete characterisation of h-vectors (and therefore, f-vectors) of Cohen–Macaulay complexes.

The following fundamental result gives a combinatorial characterisation of Cohen–Macaulay complexes:

Theorem 3.3.8 (Reisner [**172**])**.** *Let $\mathbf{k}=\mathbb{Z}$ or a field. A simplicial complex $\mathcal{K}$ is Cohen–Macaulay over $\mathbf{k}$ if and only if for any simplex $I\in\mathcal{K}$ (including $I=\varnothing$) and $i<\dim(\operatorname{lk}I)$, we have $\widetilde{H}^i(\operatorname{lk}I;\mathbf{k})=0$.*

The proof can be found in many texts on combinatorial commutative algebra, see e.g. [**184**, §II.4] or [**141**, Ch. 13]. It is not very hard, but uses the notion of *local cohomology*, which is beyond the scope of this book.

The following reformulation of Reisner's Theorem shows that the Cohen–Macaulayness is a topological property of a simplicial complex.

Proposition 3.3.9 (Munkres, Stanley)**.** *A simplicial complex $\mathcal{K}$ is Cohen–Macaulay over $\mathbf{k}$ if and only if for any point $x\in|\mathcal{K}|$, we have*

$$H^i(|\mathcal{K}|,|\mathcal{K}|\setminus x;\mathbf{k})=\widetilde{H}^i(\mathcal{K};\mathbf{k})=0\quad \textit{for } i<\dim\mathcal{K}.$$

PROOF. If $I=\varnothing$, then $\widetilde{H}^i(\mathcal{K};\mathbf{k})=\widetilde{H}^i(\operatorname{lk} I;\mathbf{k})$. If $I\neq\varnothing$, then

$$H^i(|\mathcal{K}|,|\mathcal{K}|\setminus x;\mathbf{k})=\widetilde{H}^{i-|I|}(\operatorname{lk} I;\mathbf{k})$$

for any x in the interior of I, by Proposition **??**.

If $\mathcal{K}$ is Cohen–Macaulay, then it is pure (Exercise 3.3.17) and therefore $\operatorname{lk} I$ is pure of dimension $\dim\mathcal{K}-|I|$ (Exercise **??**). Hence, $i<\dim\mathcal{K}$ implies that $i-|I|<\dim\operatorname{lk} I$ and $H^i(|\mathcal{K}|,|\mathcal{K}|\setminus x;\mathbf{k})=0$ by the identity above and Theorem 3.3.8.

On the other hand, if $H^i(|\mathcal{K}|,|\mathcal{K}|\setminus x;\mathbf{k})=0$ for $i<\dim\mathcal{K}$, then $\widetilde{H}^j(\operatorname{lk} I;\mathbf{k})=H^{j+|I|}(|\mathcal{K}|,|\mathcal{K}|\setminus x;\mathbf{k})=0$ for $j<\dim\operatorname{lk} I$, since $\dim\operatorname{lk} I+|I|\leqslant\dim\mathcal{K}$. Thus, $\mathcal{K}$ is Cohen–Macaulay by Theorem 3.3.8. $\square$

Corollary 3.3.10. *If a triangulation of a space X is a Cohen–Macaulay complex, then any other triangulation of X is Cohen–Macaulay as well.*

Corollary 3.3.11. *Any triangulated sphere is a Cohen–Macaulay complex.*

In particular, the h-vector of a triangulated sphere is an M-vector. This fact was used by Stanley in his generalisation of the UBT (Theorem **??**) to arbitrary sphere triangulations:

Theorem 3.3.12 (UBT for spheres)**.** *For any triangulated $(n-1)$-dimensional sphere $\mathcal{K}$ with m vertices, the h-vector $(h_0,h_1,\ldots,h_n)$ satisfies the inequalities*

$$h_i(\mathcal{K})\leqslant\tbinom{m-n+i-1}{i}.$$

Therefore, the UBT holds for triangulated spheres, that is,

$$f_i(\mathcal{K})\leqslant f_i\bigl(C^n(m)\bigr)\quad\text{for } i=1,\ldots,n-1.$$

PROOF. Let $A=A^0\oplus A^2\oplus\cdots\oplus A^{2n}$ be the algebra from the proof of Proposition 3.3.7, such that $\dim_{\mathbf{k}}A^{2i}=h_i$. In particular, $\dim_{\mathbf{k}}A^2=h_1=m-n$. Since A is generated by A^2, the number h_i cannot exceed the number of monomials of degree i in $m-n$ generators, i.e. $h_i\leqslant\binom{m-n+i-1}{i}$.

The rest follows from Lemma **??**. $\square$

Exercises.

Exercise 3.3.13. If $\mathcal{K}$ is a simplicial complex of dimension $n-1$, then $\dim\mathbf{k}[\mathcal{K}]=n$.

Exercise 3.3.14. Let $\boldsymbol{t}$ be an lsop in $\mathbf{k}[\mathcal{K}]$. Then the $\mathbf{k}$-vector space $\mathbf{k}[\mathcal{K}]/\boldsymbol{t}$ is generated by monomials v_I for $I\in\mathcal{K}$. (Hint: prove that $v_iv_I=0$ in $\mathbf{k}[\mathcal{K}]/\boldsymbol{t}$ for any $i\in[m]$ and for any *maximal* simplex $I\in\mathcal{K}$, and then use Proposition 3.1.9).

Exercise 3.3.15. A sequence $\boldsymbol{t}\subset\mathbf{k}[m]$ is $\mathbf{k}[m]$-regular for $\mathbf{k}[\mathcal{K}]$ as a $\mathbf{k}[m]$-module if and only the image of $\boldsymbol{t}$ in $\mathbf{k}[\mathcal{K}]$ is $\mathbf{k}[\mathcal{K}]$-regular.

Exercise 3.3.16. A finitely generated commutative $\mathbf{k}$-algebra is called a *complete intersection algebra* if it is the quotient of a polynomial algebra by a regular sequence. Observe that a complete intersection algebra is Cohen–Macaulay. Show that a face ring $\mathbf{k}[\mathcal{K}]$ is a complete intersection algebra if and only if it is isomorphic to the quotient of the form

$$\mathbf{k}[v_1,\ldots,v_m]/(v_1v_2\cdots v_{k_1},v_{k_1+1}v_{k_1+2}\cdots v_{k_1+k_2},\ldots,v_{k_1+\cdots+k_{p-1}+1}\cdots v_{k_1+\cdots+k_p}).$$

This is equivalent to $\mathcal{K}$ being decomposable into the join of the form

$$\partial\Delta^{k_1-1}*\partial\Delta^{k_2-1}*\cdots*\partial\Delta^{k_p-1}*\Delta^{m-s-1},$$

where $s=k_1+\cdots+k_p$ and the join factor Δ^{m-s-1} is void if $s=m$.

Exercise 3.3.17. A Cohen–Macaulay complex is pure. (Hint: given a maximal simplex $J \in \mathcal{K}$, consider the ideal $\mathcal{I}_J = (v_i\colon i \notin J)$ in $\mathbf{k}[m]$ (the *associated prime ideal* of $\mathbf{k}[\mathcal{K}]$ corresponding to J), and show that

$$\operatorname{depth} \mathbf{k}[\mathcal{K}] \leqslant \dim(\mathbf{k}[m]/\mathcal{I}_J) = \dim \mathbf{k}[v_j\colon j \in J].$$

3.4. Gorenstein complexes and Dehn–Sommerville relations

Gorenstein rings are a class Cohen–Macaulay rings with a special duality property. Like in the case of Cohen–Macaulayness, simplicial complexes whose face rings are Gorenstein play an important role in combinatorial commutative algebra. In a sense, nonacyclic Gorenstein complexes provide a 'best possible algebraic approximation' to sphere triangulations. We review here the most important aspects of Gorenstein complexes. The proofs of the main results of this section, in particular Theorems 3.4.2 and 3.4.3, require considerably more commutative algebraic techniques than those from the previous sections. We refer the reader to [**32**, Ch.3] for the general theory of Gorenstein rings, within which these proofs can be achieved.

We recall from Proposition 3.3.5 that nonzero Betti numbers of a Cohen–Macaulay complex $\mathcal{K}$ of dimension $n-1$ with m vertices appear up to homological degree $-(m-n)$, and $\beta^{-(m-n)}(\mathbf{k}[\mathcal{K}]) \neq 0$.

Definition 3.4.1. A Cohen–Macaulay complex $\mathcal{K}$ of dimension $n-1$ with m vertices is called *Gorenstein* (over a field $\mathbf{k}$) if $\beta^{-(m-n)}(\mathbf{k}[\mathcal{K}]) = 1$, that is, if $\operatorname{Tor}^{-(m-n)}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \mathbf{k}$. Furthermore, $\mathcal{K}$ is *Gorenstein** if $\mathcal{K}$ is Gorenstein and $\mathcal{K} = \operatorname{core} \mathcal{K}$ (see Definition **??**).

Since $\mathcal{K} = \operatorname{core}(\mathcal{K}) * \Delta^{s-1}$ for some s, we have $\mathbf{k}[\mathcal{K}] = \mathbf{k}[\operatorname{core}(\mathcal{K})] \otimes \mathbf{k}[s]$. Then Corollary A.3.3 implies that

$$\operatorname{Tor}^{-i}_{\mathbf{k}[m]}\big(\mathbf{k}[\mathcal{K}], \mathbf{k}\big) \cong \operatorname{Tor}^{-i}_{\mathbf{k}[m-s]}\big(\mathbf{k}[\operatorname{core} \mathcal{K}], \mathbf{k}\big).$$

Therefore, $\mathcal{K}$ is Gorenstein if and only if $\operatorname{core} \mathcal{K}$ is Gorenstein*.

Like in the case of Cohen–Macaulay complexes, Gorenstein* complexes can be characterised topologically as follows.

Theorem 3.4.2 ([**184**, §II.5] or [**32**, Th. 5.6.1])**.** *The following conditions are equivalent:*

(a) *$\mathcal{K}$ is a Gorenstein* complex over* $\mathbf{k}$*;*
(b) *for any simplex $I \in \mathcal{K}$ (including $I = \varnothing$) the subcomplex* $\operatorname{lk} I$ *has homology of a sphere of dimension* $\dim(\operatorname{lk} I)$*;*
(c) *for any $x \in |\mathcal{K}|$,*

$$H^i\big(|\mathcal{K}|, |\mathcal{K}| \setminus x; \mathbf{k}\big) = \widetilde{H}^i(\mathcal{K}; \mathbf{k}) = \begin{cases} \mathbf{k} & \textit{if } \ i = \dim \mathcal{K}; \\ 0 & \textit{otherwise.} \end{cases}$$

In topology, polyhedrons $|\mathcal{K}|$ satisfying the conditions of the previous theorem are sometimes called *generalised homology spheres* ('generalised' because a homology sphere is usually assumed to be a manifold). In particular, triangulated spheres are Gorenstein* complexes. Triangulated manifolds are not Gorenstein* or even Cohen–Macaulay in general (*Buchsbaum complexes* provide a proper algebraic approximation to triangulated manifolds, see [**184**, § II.8]). Nevertheless, the Tor-algebra of a Gorenstein* complex behaves like the cohomology algebra of a manifold: it has the *Poincaré duality*. This fundamental result was proved by

Avramov and Golod for Nötherian local rings; here we state the graded version of their theorem in the case of face rings.

Recall that a graded commutative connected $\mathbf{k}$-algebra A is called a *Poincaré algebra* if it is finite dimensional over $\mathbf{k}$, i.e. $A = \bigoplus_{i=0}^{d} A^i$, and the $\mathbf{k}$-linear maps

$$\begin{aligned} A^i &\to \mathrm{Hom}_{\mathbf{k}}(A^{d-i}, A^d),\\ a &\mapsto \varphi_a, \quad \text{where } \varphi_a(b) = ab \end{aligned}$$

are isomorphisms for $0 \leqslant i \leqslant d$. The classical example of a Poincaré algebra is the cohomology algebra of a manifold.

Theorem 3.4.3 (Avramov–Golod, [**32**, Th. 3.4.5])**.** *A simplicial complex $\mathcal{K}$ is Gorenstein* if and only if the algebra $T = \bigoplus_{i=0}^{d} T^i$, where $T^i = \mathrm{Tor}^{-i}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ and $d = \max\{j\colon \mathrm{Tor}^{-j}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \neq 0\}$, is a Poincaré algebra.*

Corollary 3.4.4. *Let $\mathcal{K}$ be a Gorensten* complex of dimension $n-1$ on the set $[m]$. Then the Betti numbers and the Poincaré series of the* Tor *groups satisfy*

$$\begin{aligned} \beta^{-i,\,2j}\bigl(\mathbf{k}[\mathcal{K}]\bigr) &= \beta^{-(m-n)+i,\,2(m-j)}\bigl(\mathbf{k}[\mathcal{K}]\bigr), \quad 0 \leqslant i \leqslant m-n,\ 0 \leqslant j \leqslant m,\\ F\bigl(\mathrm{Tor}^{-i}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k});\ \lambda\bigr) &= \lambda^{2m} F\bigl(\mathrm{Tor}^{-(m-n)+i}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k});\ \tfrac{1}{\lambda}\bigr). \end{aligned}$$

Proof. Theorems 3.2.3 and 3.4.2 imply that

$$\beta^{-(m-n)}\bigl(\mathbf{k}[\mathcal{K}]\bigr) = \beta^{-(m-n),2m}\bigl(\mathbf{k}[\mathcal{K}]\bigr) = 1.$$

We therefore have $d = m-n$ and $T^d = \mathrm{Tor}^{-(m-n),\,2m}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k}) \cong \mathbf{k}$ in the notation of Theorem 3.4.3. Since the multiplication in the Tor-algebra preserves the bigrading, the isomorphisms $T^i \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{k}}(T^{m-n-i}, T^{m-n})$ from the definition of a Poincaré algebra refine to the isomorphisms

$$T^{i,\,2j} \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{k}}(T^{m-n-i,\,2(m-j)}, T^{m-n,\,2m}),$$

where $T^{m-n,\,2m} \cong \mathbf{k}$. This implies the first identity, and the second is a direct corollary. $\square$

As a further corollary we obtain the following symmetry property for the Poincaré series of the face ring:

Corollary 3.4.5. *If $\mathcal{K}$ is Gorenstein* of dimension $n-1$, then*

$$F\bigl(\mathbf{k}[\mathcal{K}], \lambda\bigr) = (-1)^n F\bigl(\mathbf{k}[\mathcal{K}], \tfrac{1}{\lambda}\bigr).$$

Proof. We apply Proposition A.2.1 to the minimal resolution of the $\mathbf{k}[m]$-module $\mathbf{k}[\mathcal{K}]$. Note that $F(\mathbf{k}[m]; \lambda) = (1-\lambda^2)^{-m}$. It follows from the formula for the Poincaré series from Proposition A.2.1 and Proposition A.2.5 that

$$F\bigl(\mathbf{k}[\mathcal{K}]; \lambda\bigr) = (1-\lambda^2)^{-m} \sum_{i=0}^{m-n} (-1)^i F\Bigl(\mathrm{Tor}^{-i}_{\mathbf{k}[m]}\bigl(\mathbf{k}[\mathcal{K}], \mathbf{k}\bigr); \lambda\Bigr).$$

Using Corollary 3.4.4, we calcualte

$$F\big(\mathbf{k}[\mathcal{K}];\lambda\big)=(1-\lambda^2)^{-m}\sum_{i=0}^{m-n}(-1)^i\lambda^{2m}F\Big(\operatorname{Tor}_{\mathbf{k}[m]}^{-(m-n)+i}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big);\tfrac{1}{\lambda}\Big)=$$

$$=\big(1-(\tfrac{1}{\lambda})^2\big)^{-m}(-1)^m\sum_{j=0}^{m-n}(-1)^{m-n-j}F\Big(\operatorname{Tor}_{\mathbf{k}[m]}^{-j}\big(\mathbf{k}[\mathcal{K}],\mathbf{k}\big);\tfrac{1}{\lambda}\Big)=$$

$$=(-1)^nF\big(\mathbf{k}[\mathcal{K}];\tfrac{1}{\lambda}\big).$$

□

Corollary 3.4.6. *The Dehn–Sommerville relations $h_i=h_{n-i}$ hold for any Gorenstein* complex of dimension $n-1$ (in particular, for any triangulation of an $(n-1)$-sphere).*

Proof. This follows from the explicit form of the Poincaré series for $\mathbf{k}[\mathcal{K}]$ (Theorem 3.1.10) and the previous corollary. □

The Dehn–Sommerville relations may be further generalised to wider classes of complexes and posets; we give some of these generalisations in Section 3.8 below.

Unlike the situation with Cohen–Macaulay complexes, a complete characterisation of h-vectors (or, equivalently, f-vectors) of Gorenstein complexes is not known:

Problem 3.4.7 (Stanley)**.** Characterise the h-vectors of Gorenstein complexes.

If the *g-conjecture* (i.e. the inequalities of Theorem **??** (b) and (c)) holds for Gorenstein complexes, then this would imply a solution to the above problem.

3.4.1. Exercises.

Exercise 3.4.8. A Gorenstein complex $\mathcal{K}$ is Gorenstein* if and only if it is *nonacyclic* (i.e. $\widetilde{H}^*(\mathcal{K};\mathbf{k})\neq 0$).

Exercise 3.4.9. Let $\mathcal{K}$ be a Gorenstein* complex of dimension $n-1$ on $[m]$. Show that

$$\widetilde{H}^k(\mathcal{K}_J)\cong\widetilde{H}^{n-2-k}(\mathcal{K}_{\widehat{J}})$$

for any $J\subset[m]$, where $\widehat{J}=[m]\setminus J$. This is known as the *Alexander duality for nonacyclic Gorenstein complexes.*

3.5. Face rings of simplicial posets

The whole theory of face rings may be extended to simplicial posets, thereby leading to new important classes of rings in combinatorial commutative algebra and applications in toric topology.

The *face ring* $\mathbf{k}[\mathcal{S}]$ of a simplicial poset $\mathcal{S}$ was introduced by Stanley **[182]** as a quotient of certain graded polynomial ring by a homogeneous ideal determined by the poset relation in $\mathcal{S}$ (see Definition 3.5.2 below). The rings $\mathbf{k}[\mathcal{S}]$ have remarkable algebraic and homological properties, albeit they are much more complicated than the Stanley–Reisner face rings $\mathbf{k}[\mathcal{K}]$. Unlike $\mathbf{k}[\mathcal{K}]$, the ring $\mathbf{k}[\mathcal{S}]$ is not generated in the lowest positive degree. Face rings of simplicial posets were further studied by Duval **[74]** and Maeda–Masuda–Panov **[131]**, **[127]**, among others. *Cohen–Macaulay* and *Gorenstein** face rings are particularly important; both properties

are topological, that is, depend only on the topological type of the geometric realisation $|\mathcal{S}|$.

As usual, we shall not distinguish between simplicial posets $\mathcal{S}$ and their geometric realisations (simplicial cell complexes) $|\mathcal{S}|$. Given two elements $\sigma, \tau \in \mathcal{S}$, we denote by $\sigma \vee \tau$ the set of their joins, and denote by $\sigma \wedge \tau$ the set of their *meets*. Whenever either of these sets consists of a single element, we use the same notation for this particular element of $\mathcal{S}$.

To make clear the idea behind the definition of the face ring of a simplicial poset, we first consider the case when $\mathcal{S}$ is a simplicial complex $\mathcal{K}$. Then $\sigma \wedge \tau$ consists of a single element (possibly $\varnothing$), and $\sigma \vee \tau$ is either empty or consists of a single element. We consider the graded polynomial ring $\mathbf{k}[v_\sigma \colon \sigma \in \mathcal{K}]$ with one generator v_σ of degree $\deg v_\sigma = 2|\sigma|$ for each simplex $\sigma \in \mathcal{K}$. The following proposition provides an alternative presentation of the face ring $\mathbf{k}[\mathcal{K}]$, with a larger set of generators:

Proposition 3.5.1. *There is a canonical isomorphism of graded rings*

$$\mathbf{k}[\mathcal{K}] \cong \mathbf{k}[v_\sigma \colon \sigma \in \mathcal{K}]/\mathcal{I}'_{\mathcal{K}},$$

where $\mathcal{I}'_{\mathcal{K}}$ is the ideal generated by the element $v_\varnothing - 1$ and all elements of the form

$$v_\sigma v_\tau - v_{\sigma\wedge\tau} v_{\sigma\vee\tau}.$$

Here we set $v_{\sigma\vee\tau} = 0$ whenever $\sigma \vee \tau$ is empty.

Proof. The isomorphism is established by the map taking v_σ to $\prod_{i\in\sigma} v_i$. The rest is left as an exercise. □

Now let $\mathcal{S}$ be an arbitrary simplicial poset with the vertex set $V(\mathcal{S}) = [m]$. In this case both $\sigma \vee \tau$ and $\sigma \wedge \tau$ may consist of more than one element, but $\sigma \wedge \tau$ consists of a single element whenever $\sigma \vee \tau$ is nonempty.

We consider the graded polynomial ring $\mathbf{k}[v_\sigma \colon \sigma \in \mathcal{S}]$ with one generator v_σ of degree $\deg v_\sigma = 2|\sigma|$ for every element $\sigma \in \mathcal{S}$.

Definition 3.5.2 (**[182]**)**.** The *face ring* of a simplicial poset $\mathcal{S}$ is the quotient

$$\mathbf{k}[\mathcal{S}] = \mathbf{k}[v_\sigma \colon \sigma \in \mathcal{S}] \,/\, \mathcal{I}_{\mathcal{S}},$$

where $\mathcal{I}_{\mathcal{S}}$ is the ideal generated by the elements $v_{\hat{0}} - 1$ and

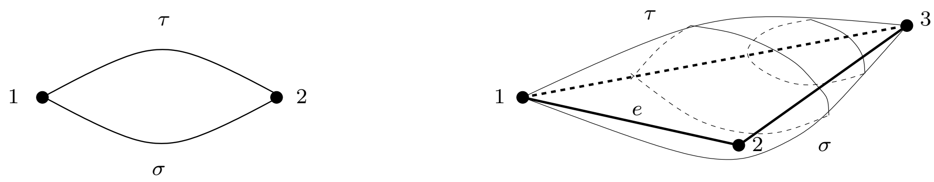
$$v_\sigma v_\tau - v_{\sigma\wedge\tau} \cdot \sum_{\eta\in\sigma\vee\tau} v_\eta. \tag{3.10}$$

The sum over the empty set is assumed to be zero, so we have $v_\sigma v_\tau = 0$ in $\mathbf{k}[\mathcal{S}]$ if $\sigma \vee \tau$ is empty.

The grading may be refined to an $\mathbb{N}^m$-grading by setting $\operatorname{mdeg} v_\sigma = 2V(\sigma)$. Here $V(\sigma) \subset [m]$ is the vertex set of σ, and we identify subsets of $[m]$ with $(0,1)$-vectors in $\{0,1\}^m \subset \mathbb{N}^m$ as usual. In particular, $\operatorname{mdeg} v_i = 2\boldsymbol{e}_i$ (two times the ith basis vector).

Example 3.5.3. 1. The simplicial cell complex shown in Fig. 3.3 (a) is obtained by gluing two segments along their boundaries and has rank 2. The vertices are $1, 2$ and we denote the 1-dimensional simplices by σ and τ. Then the face ring $\mathbf{k}[\mathcal{S}]$ is the quotient of the graded polynomial ring

$$\mathbf{k}[v_1, v_2, v_\sigma, v_\tau], \quad \deg v_1 = \deg v_2 = 2, \quad \deg v_\sigma = \deg v_\tau = 4$$



(a) $r = 2$ (b) $r = 3$

Figure 3.3. Simplicial cell complexes.

by the two relations

$$v_1v_2 = v_\sigma + v_\tau, \quad v_\sigma v_\tau = 0.$$

2. The simplicial cell complex in Fig. 3.3 (b) is obtained by gluing two triangles along their boundaries and has rank 3. The vertices are $1, 2, 3$ and we denote the 1-dimensional simplices (edges) by e, f and g, and the 2-dimensional simplices by σ and τ. The face ring $\mathbf{k}[\mathcal{S}]$ is isomorphic to the quotient of the graded polynomial ring

$$\mathbf{k}[v_1, v_2, v_3, v_\sigma, v_\tau], \quad \deg v_1 = \deg v_2 = \deg v_3 = 2, \quad \deg v_\sigma = \deg v_\tau = 6$$

by the two relations

$$v_1v_2v_3 = v_\sigma + v_\tau, \quad v_\sigma v_\tau = 0.$$

The generators corresponding to the edges can be excluded because of the relations $v_e = v_1v_2$, $v_f = v_2v_3$ and $v_g = v_1v_3$.

Remark. The ideal $\mathcal{I}_\mathcal{S}$ is generated by *straightening relations* (3.10), in the sense that these relations allow us to express the product of any pair of generators via products of generators corresponding to pairs of ordered elements of the poset. This can be restated by saying that $\mathbf{k}[\mathcal{S}]$ is an example of an *algebra with straightening law* (ASL for short, also known as a *Hodge algebra*). Lemma 3.5.4 and Theorem 3.5.7 below reflect algebraic properties of ASL's, and may be restated in this generality. For more on the theory of ASL's see [**184**, § III.6] and [**32**, Ch. 7].

A monomial $v_{\sigma_1}^{i_1}v_{\sigma_2}^{i_2}\cdots v_{\sigma_k}^{i_k} \in \mathbf{k}[v_\sigma \colon \sigma \in \mathcal{S}]$ is *standard* if $\sigma_1 < \sigma_2 < \cdots < \sigma_k$.

Lemma 3.5.4. *Any element of* $\mathbf{k}[\mathcal{S}]$ *can be written as a linear combination of standard monomials.*

Proof. It is enough to prove the statement for elements of $\mathbf{k}[\mathcal{S}]$ represented by monomials in generators v_σ. We write such a monomial as $a = v_{\tau_1}v_{\tau_2}\cdots v_{\tau_k}$ where some of the τ_i may coincide. We need to show that any such monomial can be expressed as a sum of monomials $\sum v_{\sigma_1}\cdots v_{\sigma_l}$ with $\sigma_1 \leqslant \cdots \leqslant \sigma_l$. We may assume by induction that $\tau_2 \leqslant \cdots \leqslant \tau_k$. Using relation (3.10) we can replace a by

$$v_{\tau_1\wedge\tau_2}\Bigl(\sum_{\rho\in\tau_1\vee\tau_2} v_\rho\Bigr)v_{\tau_3}\cdots v_{\tau_k}.$$

Now the first two factors in each summand above correspond to ordered elements of $\mathcal{S}$. We proceed by replacing the products $v_\rho v_{\tau_3}$ by $v_{\rho\wedge\tau_3}(\sum_{\pi\in\rho\vee\tau_3} v_\pi)$. Since $\tau_1\wedge\tau_2 \leqslant \rho\wedge\tau_3$, now the first three factors in each monomial are in order. Continuing this process, we obtain in the end a sum of monomials corresponding to totally ordered sets of elements of $\mathcal{S}$. $\square$

We refer to the presentation from Lemma 3.5.4 as a *standard representation* of an element $a \in \mathbf{k}[\mathcal{S}]$.

Given $\sigma \in \mathcal{S}$, we define the corresponding *restriction homomorphism* as

$$s_\sigma \colon \mathbf{k}[\mathcal{S}] \to \mathbf{k}[\mathcal{S}]/(v_\tau \colon \tau \nleqslant \sigma).$$

The following result is straightforward.

Proposition 3.5.5. *Let $|\sigma| = k$ with $V(\sigma) = \{i_1, \ldots, i_k\}$. Then the image of the homomorphism s_σ is the polynomial ring $\mathbf{k}[v_{i_1}, \ldots, v_{i_k}]$.*

The next result generalises Proposition 3.1.8 to simplicial posets.

Theorem 3.5.6. *The direct sum*

$$s = \textstyle\bigoplus_{\sigma \in \mathcal{S}} s_\sigma \colon \mathbf{k}[\mathcal{S}] \longrightarrow \displaystyle\bigoplus_{\sigma \in \mathcal{S}} \mathbf{k}[v_i \colon i \in V(\sigma)]$$

of all restriction maps is a monomorphism.

Proof. Take a nonzero element $a \in \mathbf{k}[\mathcal{S}]$ and write its standard representation. Fix a standard monomial $v_{\sigma_1}^{i_1} \cdots v_{\sigma_k}^{i_k}$ which enters this decomposition with a nonzero coefficient. Allowing some of the exponents i_j to be zero, we may assume that σ_k is a maximal element in $\mathcal{S}$ and $|\sigma_j| = j$ for $1 \leqslant j \leqslant k$. We shall prove that $s_{\sigma_k}(a) \neq 0$. Identify $s_{\sigma_k}(\mathbf{k}[\mathcal{S}])$ with the polynomial ring $\mathbf{k}[t_1, \ldots, t_k]$ (so that $t_j = v_{i_j}$ in the notation of Proposition 3.5.5). Then $s_{\sigma_k}(v_{\sigma_k}) = t_1 \cdots t_k$, and we may assume without loss of generality that $s_{\sigma_k}(v_{\sigma_j}) = t_1 \cdots t_j$ for $1 \leqslant j \leqslant k$. Hence,

$$s_{\sigma_k}\bigl(v_{\sigma_1}^{i_1} \cdots v_{\sigma_k}^{i_k}\bigr) = t_1^{i_1}(t_1 t_2)^{i_2} \cdots (t_1 \cdots t_k)^{i_k}.$$

If we prove that no other monomial $v_{\tau_1}^{j_1} \cdots v_{\tau_m}^{j_m}$ is mapped by s_{σ_k} to the same element of $\mathbf{k}[t_1, \ldots, t_k]$, then this would imply that $s_{\sigma_k}(a) \neq 0$. Note that

$$s_{\sigma_k}(v_{\tau_1}^{j_1} \cdots v_{\tau_m}^{j_m}) = 0 \qquad \text{if } \tau_i \nleqslant \sigma_k \text{ for some } i \text{ with } j_i \neq 0,$$

so that we may assume that $m = k$. Now suppose that

$$s_{\sigma_k}\bigl(v_{\sigma_1}^{i_1} \cdots v_{\sigma_k}^{i_k}\bigr) = s_{\sigma_k}\bigl(v_{\tau_1}^{j_1} \cdots v_{\tau_k}^{j_k}\bigr). \tag{3.11}$$

We shall prove that $v_{\sigma_1}^{i_1} \cdots v_{\sigma_k}^{i_k} = v_{\tau_1}^{j_1} \ldots v_{\tau_k}^{j_k}$. We may assume by induction that the 'tails' of these monomial coincide, that is, there is some q, $1 \leqslant q \leqslant k$, such that $i_p = j_p$ and $\sigma_p = \tau_p$ for $i_p \neq 0$ whenever $p > q$. We shall prove that $i_q = j_q$ and $\sigma_q = \tau_q$ if $i_q \neq 0$. We obtain from (3.11) that

$$\begin{aligned} s_{\sigma_k}\bigl(v_{\sigma_1}^{i_1} \cdots v_{\sigma_q}^{i_q}\bigr)&(t_1 \cdots t_{q+1})^{i_{q+1}} \cdots (t_1 \cdots t_k)^{i_k} = \\ &= s_{\sigma_k}\bigl(v_{\tau_1}^{j_1} \cdots v_{\tau_q}^{j_q}\bigr)(t_1 \cdots t_{q+1})^{i_{q+1}} \cdots (t_1 \cdots t_k)^{j_k}, \end{aligned}$$

hence, $s_{\sigma_k}(v_{\sigma_1}^{i_1} \cdots v_{\sigma_q}^{i_q}) = s_{\sigma_k}(v_{\tau_1}^{j_1} \cdots v_{\tau_q}^{j_q})$. Let j_l be the last nonzero exponent in $v_{\tau_1}^{j_1} \cdots v_{\tau_q}^{j_q}$ (i.e. $j_{l+1} = \cdots = j_q = 0$). Then we also have $i_{l+1} = \cdots = i_q = 0$, as otherwise $s_{\sigma_k}(v_{\sigma_1}^{i_1} \cdots v_{\sigma_q}^{i_q})$ is divisible by $t_1 \cdots t_{l+1}$, while $s_{\sigma_k}(v_{\tau_1}^{j_1} \cdots v_{\tau_q}^{j_q})$ is not. We also have $i_l = j_l$ and $\sigma_l = \tau_l$, since i_l is the maximal power of the monomial $t_1 \cdots t_l$ which divides $s_{\sigma_k}(v_{\sigma_1}^{i_1} \cdots v_{\sigma_q}^{i_q})$. We conclude by induction that $v_{\sigma_1}^{i_1} \cdots v_{\sigma_k}^{i_k} = v_{\tau_1}^{j_1} \cdots v_{\tau_k}^{j_k}$, and $s_{\sigma_k}(a) \neq 0$. □

Remark. The proof above also shows that the map $s = \bigoplus_\sigma s_\sigma$ in Theorem 3.5.6 can be defined as the sum over the maximal elements $\sigma \in \mathcal{S}$ only.

Theorem 3.5.7. *The standard representation of an element $a \in \mathbf{k}[\mathcal{S}]$ is unique. In other words, the standard monomials $v_{\sigma_1}^{i_1}\cdots v_{\sigma_k}^{i_k}$ form a $\mathbf{k}$-basis of $\mathbf{k}[\mathcal{S}]$.*

PROOF. This follows directly from Lemma 3.5.4 and Theorem 3.5.6. □

Lemma 3.3.1, which describes hsop's in the face rings of simplicial complexes, can be readily extended to simplicial posets (the same proof based on the properties of the restriction map s works):

Lemma 3.5.8. *Let $\mathcal{S}$ be a simplicial poset of rank n. A sequence of homogeneous elements $\boldsymbol{t} = (t_1, \dots, t_n)$ of $\mathbf{k}[\mathcal{S}]$ is a homogeneous system of parameters if and only*

$$\dim_{\mathbf{k}}\bigl(\mathbf{k}[v_i\colon i \in V(\sigma)]/s_\sigma(\boldsymbol{t})\bigr) < \infty$$

for each element $\sigma \in \mathcal{S}$.

Define the *f-vector* of a simplicial poset $\mathcal{S}$ as $\boldsymbol{f}(\mathcal{S}) = (f_0, \dots, f_{n-1})$, where $n-1 = \dim \mathcal{S}$ and f_i is the number of elements of rank $i+1$ (i.e. number of faces of dimension i in the simplicial cell complex). As usual, the *h-vector* $\boldsymbol{h}(\mathcal{S}) = (h_0, \dots, h_n)$ is defined by (**??**).

The Poincaré series of the face ring $\mathbf{k}[\mathcal{S}]$ has exactly the same form as in the case of simplicial complexes:

Theorem 3.5.9. *We have*

$$F\bigl(\mathbf{k}[\mathcal{S}];\lambda\bigr) = \sum_{k=0}^{n} \frac{f_{k-1}\lambda^{2k}}{(1-\lambda^2)^k} = \frac{h_0 + h_1\lambda^2 + \cdots + h_n\lambda^{2n}}{(1-\lambda^2)^n}.$$

PROOF. By Theorem 3.5.7, we need to calculate the Poincaré series of the $\mathbf{k}$-vector space generated by the monomials $v_{\sigma_1}^{i_1}\cdots v_{\sigma_k}^{i_k}$ with $\sigma_1 < \cdots < \sigma_k$. For every $\sigma \in \mathcal{S}$ denote by $\mathcal{M}_\sigma$ the set of such monomials with $\sigma_k = \sigma$ and $i_k > 0$. Let $|\sigma| = k$; consider the restriction homomorphism s_σ to the polynomial ring $\mathbf{k}[t_1, \dots, t_k]$. Then $s_\sigma(\mathcal{M}_\sigma)$ is the set of monomials in $\mathbf{k}[t_1, \dots, t_k]$ which are divisible by $t_1 \cdots t_k$. Therefore, the Poincaré series of the subspace generated by the set $\mathcal{M}_\sigma$ is $\frac{\lambda^{2k}}{(1-\lambda^2)^k}$. Now, to finish the proof of the first identity we note that $\mathcal{S}$ is the union $\cup_{\sigma\in\mathcal{S}}\mathcal{M}_\sigma$ of the nonintersecting subsets $\mathcal{M}_\sigma$. The second identity follows from (**??**). □

As we have seen in Exercise 3.1.15, the face ring $\mathbf{k}[\mathcal{K}]$ of a simplicial complex can be realised as the limit of a diagram of polynomial algebras over $\mathrm{CAT}^{op}(\mathcal{K})$. A similar description exists for the face ring $\mathbf{k}[\mathcal{S}]$:

Construction 3.5.10 ($\mathbf{k}[\mathcal{S}]$ as limit)**.** We consider the diagram $\mathbf{k}[\cdot]_{\mathcal{S}}$ similar to that of Exercise 3.1.15:

$$\begin{aligned} \mathbf{k}[\cdot]_{\mathcal{S}}\colon \mathrm{CAT}^{op}(\mathcal{S}) &\longrightarrow \mathrm{CGA}, \\ \sigma &\longmapsto \mathbf{k}[v_i\colon i \in V(\sigma)], \end{aligned}$$

whose value on a morphism $\sigma \leqslant \tau$ is the surjection

$$\mathbf{k}[v_i\colon i \in V(\tau)] \to \mathbf{k}[v_i\colon i \in V(\sigma)]$$

sending each v_i with $i \notin V(\sigma)$ to zero.

Lemma 3.5.11. *We have*

$$\mathbf{k}[\mathcal{S}] = \lim \mathbf{k}[\cdot]_{\mathcal{S}}$$

where the limit is taken in the category CGA.

PROOF. We set up a total order on the elements of $\mathcal{S}$ so that the rank function does not decrease, and proceed by induction. We therefore may assume the statement is proved for a simplicial poset $\mathcal{T}$, and need to prove it for $\mathcal{S}$ which is obtained from $\mathcal{T}$ by adding one element σ. Then $\mathcal{S}_{<\sigma} = \{\tau \in \mathcal{S} \colon \tau < \sigma\}$ is the face poset of the boundary of the simplex Δ^σ. Geometrically, we may think of $|\mathcal{S}|$ as obtained from $|\mathcal{T}|$ by attaching one simplex Δ^σ along its boundary (if $|\sigma| = 1$, then Δ^σ is a single point, so $|\mathcal{S}|$ is a disjoint union of $|\mathcal{T}|$ and a point). We therefore need to prove that the following is a pullback diagram:

$$\begin{array}{ccc} \mathbf{k}[\mathcal{S}] & \longrightarrow & \mathbf{k}[\mathcal{S}_{\leqslant\sigma}] \\ \big\downarrow & & \big\downarrow \\ \mathbf{k}[\mathcal{T}] & \longrightarrow & \mathbf{k}[\mathcal{S}_{<\sigma}]. \end{array} \tag{3.12}$$

Here the vertical arrows map v_σ to 0, while the horizontal ones map v_τ to 0 for $\tau \nleqslant \sigma$. Denote by A the pullback of (3.12) with $\mathbf{k}[\mathcal{S}]$ dropped. We need to show that the natural map $\mathbf{k}[\mathcal{S}] \to A$ is an isomorphism.

Since the limits in CGA are created in the underlying category of graded $\mathbf{k}$-vector spaces, the space of A is the direct sum of $\mathbf{k}[\mathcal{T}]$ and $\mathbf{k}[\mathcal{S}_{\leqslant\sigma}]$ with the pieces $\mathbf{k}[\mathcal{S}_{<\sigma}]$ identified in both spaces. In other words,

$$A = T \oplus \mathbf{k}[\mathcal{S}_{<\sigma}] \oplus S, \tag{3.13}$$

where T is the complement to $\mathbf{k}[\mathcal{S}_{<\sigma}]$ in $\mathbf{k}[\mathcal{T}]$, and S is the complement to $\mathbf{k}[\mathcal{S}_{<\sigma}]$ in $\mathbf{k}[\mathcal{S}_{\leqslant\sigma}]$. By Theorem 3.5.7, the space $\mathbf{k}[\mathcal{S}_{<\sigma}]$ has basis of standard monomials $v_{\tau_1}^{j_1} v_{\tau_2}^{j_2} \cdots v_{\tau_k}^{j_k}$ with $\tau_k < \sigma$. Similarly, S has basis of those monomials with $\tau_k = \sigma$ and $j_k > 0$, while T has basis of those monomials with $\tau_k \nleqslant \sigma$ and $j_k > 0$. Yet another application of Theorem 3.5.7 gives a decomposition of $\mathbf{k}[\mathcal{S}]$ identical to (3.13): a standard basis monomial $v_{\tau_1}^{j_1} v_{\tau_2}^{j_2} \cdots v_{\tau_k}^{j_k}$ with $j_k > 0$ has either $\tau_k \nleqslant \sigma$, or $\tau_k < \sigma$, or $\tau_k = \sigma$. These three possibilities map to T, $\mathbf{k}[\mathcal{S}_{<\sigma}]$ and S respectively. It follows that $\mathbf{k}[\mathcal{S}] \to A$ is an isomorphism of $\mathbf{k}$-vector spaces. Since it is an algebra map, it is also an isomorphism of algebras, thus finishing the proof. □

The description of $\mathbf{k}[\mathcal{S}]$ as a limit has the following important corollary, describing the functorial properties of the face rings and generalising Proposition 3.1.5.

Proposition 3.5.12. *Let $f \colon \mathcal{S} \to \mathcal{T}$ be a rank-preserving map of simplicial posets. Define a homomorphism*

$$f^* \colon \mathbf{k}[w_\tau \colon \tau \in \mathcal{T}] \to \mathbf{k}[v_\sigma \colon \sigma \in \mathcal{S}], \quad f^*(w_\tau) = \sum_{\sigma \in f^{-1}(\tau),\ |\sigma| = |\tau|} v_\sigma.$$

Then f^* descends to a ring homomorphism $\mathbf{k}[\mathcal{T}] \to \mathbf{k}[\mathcal{S}]$, which we continue to denote by f^*.

PROOF. The poset map f gives rise to a functor $f \colon \mathrm{CAT}^{op}(\mathcal{S}) \to \mathrm{CAT}^{op}(\mathcal{T})$ and therefore to a natural transformation

$$f^* \colon [\mathrm{CAT}^{op}(\mathcal{T}), \mathrm{CGA}] \to [\mathrm{CAT}^{op}(\mathcal{S}), \mathrm{CGA}],$$

where $[\mathrm{CAT}^{op}(\mathcal{S}), \mathrm{CGA}]$ denotes the functors from $\mathrm{CAT}^{op}(\mathcal{S})$ to CGA. It is easy to see that $f^*\mathbf{k}[\cdot]_\mathcal{T} = \mathbf{k}[\cdot]_\mathcal{S}$ in the notation of Construction 3.5.10, so we have the induced map of limits $f^* \colon \mathbf{k}[\mathcal{T}] \to \mathbf{k}[\mathcal{S}]$. We also have that $f^*(w_\tau) = \sum_{\sigma \in f^{-1}(\tau)} v_\sigma$ by the construction of lim in CGA. □

Example 3.5.13. The folding map (**??**) induces a monomorphism $\mathbf{k}[\mathcal{K}_\mathcal{S}] \to \mathbf{k}[\mathcal{S}]$, which embeds $\mathbf{k}[\mathcal{K}_\mathcal{S}]$ in $\mathbf{k}[\mathcal{S}]$ as the subring generated by the elements v_i.

Remark. An attempt to prove Proposition 3.5.12 directly from the definition, by showing that $f^*(\mathcal{I}_\mathcal{T}) \subset \mathcal{I}_\mathcal{S}$, runs into a complicated combinatorial analysis of the poset structure. This is an example of a situation where the use of an abstract categorical description of $\mathbf{k}[\mathcal{S}]$ proves to be beneficial.

Let $\mathbf{k}[m] = \mathbf{k}[v_1, \ldots, v_m]$ be the polynomial algebra on m generators of degree 2 corresponding to the vertices of $\mathcal{S}$. The face ring $\mathbf{k}[\mathcal{S}]$ acquires a $\mathbf{k}[m]$-algebra structure via the map $\mathbf{k}[m] \to \mathbf{k}[\mathcal{S}]$ sending each v_i identically. (Unlike the case of simplicial complexes, this map is generally not surjective.) We thereby obtain a $\mathbb{Z} \oplus \mathbb{N}^m$-graded Tor-algebra of $\mathbf{k}[\mathcal{S}]$:

$$\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{S}], \mathbf{k}\big) = \bigoplus_{i \geqslant 0,\, \boldsymbol{a} \in \mathbb{N}^m} \operatorname{Tor}^{-i,\,2\boldsymbol{a}}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{S}], \mathbf{k}\big),$$

by analogy with Construction 3.2.7 for simplicial complexes.

We finish this section by stating a generalisation of Hochster's theorem to simplicial posets, and deriving some of its corollaries.

Theorem 3.5.14 (Duval [**74**], see also [**123**])**.** *For any subset $J \subset [m]$ we have*

$$\operatorname{Tor}^{-i,2J}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{S}], \mathbf{k}\big) \cong \widetilde{H}^{|J|-i-1}(|\mathcal{S}_J|; \mathbf{k}),$$

where $\mathcal{S}_J$ the subposet of $\mathcal{S}$ consisting of those σ for which $V(\sigma) \subset J$. Also, $\operatorname{Tor}^{-i,2\boldsymbol{a}}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{S}], \mathbf{k}) = 0$ if $\boldsymbol{a}$ is not a $(0,1)$-vector.

This can be proved along the lines of the proof of Theorem 3.2.8, but the construction of the chain homotopy generalising the argument of Lemma 3.2.5 to simplicial posets requires some care. We postpone the proof of Theorem 3.5.14 to Section **??**, in order to be able to make use of topological techniques developed there.

We define the *multigraded algebraic Betti numbers* of $\mathbf{k}[\mathcal{S}]$ as

$$\beta^{-i,\,2\boldsymbol{a}}\big(\mathbf{k}[\mathcal{S}]\big) = \dim_\mathbf{k} \operatorname{Tor}^{-i,\,2\boldsymbol{a}}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{S}], \mathbf{k}\big),$$

for $0 \leqslant i \leqslant m$, $\boldsymbol{a} \in \mathbb{N}^m$. We also set

$$\beta^{-i}\big(\mathbf{k}[\mathcal{S}]\big) = \dim_\mathbf{k} \operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\ldots,v_m]}\big(\mathbf{k}[\mathcal{S}], \mathbf{k}\big) = \sum_{\boldsymbol{a} \in \mathbb{N}^m} \beta^{-i,\,2\boldsymbol{a}}\big(\mathbf{k}[\mathcal{S}]\big).$$

Example 3.5.15. Let $\mathcal{S}$ be the simplicial poset of Example 3.5.3.1. By Theorem 3.5.14, $\beta^{0,(0,0)}(\mathbf{k}[\mathcal{S}]) = \beta^{0,(2,2)}(\mathbf{k}[\mathcal{S}]) = 1$, and the other Betti numbers are zero. This implies that $\mathbf{k}[\mathcal{S}]$ is a free $\mathbf{k}[v_1, v_2]$-module with two generators, 1 and v_σ, of degree 0 and 4 respectively.

Note that unlike the case of simplicial complexes, $\beta^0(\mathbf{k}[\mathcal{S}])$ may be bigger than 1. In fact, the following proposition follows easily from Theorem 3.5.14.

Proposition 3.5.16. *The number of generators of $\mathbf{k}[\mathcal{S}]$ as a $\mathbf{k}[m]$-module equals*

$$\beta^0(\mathbf{k}[\mathcal{S}]) = \sum_{J \subset [m]} \dim \widetilde{H}^{|J|-1}\big(|\mathcal{S}_J|\big).$$

3.5.1. Exercises.

Exercise 3.5.17. Finish the proof of Proposition 3.5.1.

Exercise 3.5.18. Calculate the multigraded Betti numbers for the simplicial poset of Example 3.5.3.2.

Face rings: additional topics

3.6. Cohen–Macaulay simplicial posets

Assume given a property A of simplicial complexes. Then we can extend this property to posets by postulating that a poset $\mathcal{P}$ has the property A if the order complex $\mathrm{ord}(\mathcal{P})$ (see Definition **??**) has the property A. In particular, Cohen–Macaulay and Gorenstein posets can be defined in this way. Simplicial posets $\mathcal{S}$ are of particular interest to us; in this case the order complex is identified with the barycentric subdivision $\mathcal{S}'$ (to be precise, with the cone over the barycentric subdivision, as we include the empty simplex, but this difference is inessential for the definitions to follow).

Definition 3.6.1. A simplicial poset $\mathcal{S}$ is *Cohen–Macaulay* (over $\mathbf{k}$) if its barycentric subdivision $\mathcal{S}'$ is a Cohen–Macaulay simplicial complex.

By definition, $\mathcal{S}$ is a Cohen–Macaulay simplicial poset if and only if the face ring $\mathbf{k}[\mathcal{S}']$ is Cohen–Macaulay. Since the face ring is also defined for the face poset $\mathcal{S}$ itself (and not only for its barycentric subdivision), it is perfectly natural to ask whether the class of Cohen–Macaulay simplicial posets admits an intrinsic description in terms of their face rings $\mathbf{k}[\mathcal{S}]$. One would achieve such a description by proving that the ring $\mathbf{k}[\mathcal{S}']$ is Cohen–Macaulay if and only if the ring $\mathbf{k}[\mathcal{S}]$ is Cohen–Macaulay. The 'if' part follows from the general theory of ASL's, see [**182**, Cor. 3.7]. The 'only if' part was proved in [**127**]; the proof uses the decomposition of the barycentric subdivision into a sequence of stellar subdivisions and then goes on by showing that the Cohen–Macaulay property is preserved under stellar subdivisions. We include this characterisation of Cohen–Macaulay simplicial posets in terms of their face rings in Theorem 3.6.7 below.

Since many of the construction in this section are geometric, we often talk about simplicial cell complexes rather than simplicial posets. We say that a simplicial subdivision of a simplicial cell complex $\mathcal{S}$ is *regular* if it is a simplicial complex. For instance, the barycentric subdivision is regular. Since the Cohen–Macaulayness of a simplicial complex is a topological property (see Proposition 3.3.9), we have the following statement.

Proposition 3.6.2. *The following conditions are equivalent:*

(a) *the barycentric subdivision of a simplicial cell complex $\mathcal{S}$ is a Cohen–Macaulay complex;*
(b) *any regular subdivision of $\mathcal{S}$ is a Cohen–Macaulay complex;*
(c) *a regular subdivision of $\mathcal{S}$ is a Cohen–Macaulay complex.*

As a further corollary we obtain that Proposition 3.3.9 itself extends to simplicial cell complexes, i.e. the property of a simplicial cell complex to be Cohen–Macaulay is also topological.

By analogy with Definition **??**, we define the *star* and the *link* of $\sigma \in \mathcal{S}$ as the following subcomplexes:

$$\operatorname{st}_{\mathcal{S}} \sigma = \{\tau \in \mathcal{S} \colon \sigma \vee \tau \text{ is nonempty}\};$$
$$\operatorname{lk}_{\mathcal{S}} \sigma = \{\tau \in \mathcal{S} \colon \sigma \vee \tau \text{ is nonempty, and } \tau \wedge \sigma = \hat{0}\}.$$

Remark. If $\mathcal{S}$ is a simplicial complex, then the poset $\operatorname{lk}_{\mathcal{S}} \sigma$ is isomorphic to the open semiinterval

$$\mathcal{S}_{>\sigma} = \{\rho \in \mathcal{S} \colon \rho > \sigma\},$$

and $|\operatorname{st}_{\mathcal{S}} \sigma| \cong \Delta^{\sigma} * |\operatorname{lk}_{\mathcal{S}} \sigma|$, where $*$ denotes the join. However, none of these isomorphisms holds for general $\mathcal{S}$, see Example 3.6.5 below.

Because of this remark, we cannot simply extend the definition of stellar subdivisions (Definition **??**) to simplicial cell complexes. Instead, we define the *stellar subdivision* $\operatorname{ss}_{\sigma} \mathcal{S}$ of $\mathcal{S}$ at σ as the simplicial cell complex obtained by stellarly subdividing each face containing σ in a compatible way.

Proposition 3.6.3. *The barycentric subdivision $\mathcal{S}'$ can be obtained as a sequence of stellar subdivisions, one at each face $\sigma \in \mathcal{S}$, starting from the maximal faces. Moreover, each stellar subdivision in the sequence is applied to a face whose star is a simplicial complex.*

Proof. Assume $\dim \mathcal{S} = n-1$. We start by applying to $\mathcal{S}$ stellar subdivisions at all $(n-1)$-dimensional faces. Denote the resulting complex by $\mathcal{S}_1$. The $(n-2)$-faces of $\mathcal{S}_1$ are of two types: the 'old' ones, remaining from $\mathcal{S}$, and the 'new' ones, appearing as the result of the stellar subdivisions. Then we take stellar subdivisions of $\mathcal{S}_1$ at all 'old' $(n-2)$-faces, and denote the result by $\mathcal{S}_2$. Next we apply to $\mathcal{S}_2$ stellar subdivisions at all $(n-3)$-faces remaining from $\mathcal{S}$. Proceeding in this way, at the end we get $\mathcal{S}_{n-1} = \mathcal{S}'$. To prove the second statement, consider two subsequent complexes $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ in the sequence, so that $\widetilde{\mathcal{R}}$ is obtained from $\mathcal{R}$ by a single stellar subdivision at some $\sigma \in \mathcal{S}$. Then $\operatorname{st}_{\mathcal{R}} \sigma$ is isomorphic to $\Delta^{\sigma} * (\mathcal{S}_{>\sigma})'$ and therefore it is a simplicial complex. □

We proceed with two lemmata necessary to prove our main result.

Lemma 3.6.4. *Let $\mathcal{S}$ be a simplicial poset of rank n with the vertex set $V(\mathcal{S}) = [m]$, and assume that the first k vertices span a face σ. Assume further that $\operatorname{st}_{\mathcal{S}} \sigma$ is a simplicial complex, and let $\widetilde{\mathcal{S}}$ be the stellar subdivision of $\mathcal{S}$ at σ. Let v denote the degree-two generator of $\mathbf{k}[\widetilde{\mathcal{S}}]$ corresponding to the added vertex. Then there exists a unique homomorphism $\beta \colon \mathbf{k}[\mathcal{S}] \to \mathbf{k}[\widetilde{\mathcal{S}}]$ such that*

$$\begin{aligned} v_\tau &\mapsto v_\tau && \text{for } \tau \notin \operatorname{st}_{\mathcal{S}} \sigma;\\ v_i &\mapsto v + v_i, && \text{for } i = 1, \ldots, k;\\ v_i &\mapsto v_i, && \text{for } i = k+1 \ldots, m. \end{aligned}$$

Moreover, β is a injective, and if $\boldsymbol{t}$ is an hsop in $\mathbf{k}[\mathcal{S}]$, then $\beta(\boldsymbol{t})$ is an hsop in $\mathbf{k}[\widetilde{\mathcal{S}}]$.

Proof. In order to define the map β we first need to specify the images of v_τ for all $\tau \in \operatorname{st}_{\mathcal{S}} \sigma$. Choose such a v_τ and let $V(\tau) = \{i_1, \ldots, i_\ell\}$ be its vertex set. Then we have the following identity in the ring $\mathbf{k}[\mathcal{S}] = \mathbf{k}[v_\tau \colon \tau \in \mathcal{S}]/\mathcal{I}_{\mathcal{S}}$:

$$v_{i_1} \cdots v_{i_\ell} = v_\tau + \sum_{\eta \colon V(\eta) = V(\tau),\, \eta \neq \tau} v_\eta. \tag{3.14}$$

For any v_η in the latter sum we have $\eta \notin \mathrm{st}_{\mathcal{S}}\,\sigma$, since $\mathrm{st}_{\mathcal{S}}\,\sigma$ is a simplicial complex, in which any set of vertices spans at most one face. Since β is already defined on the product on the left hand side and on the sum on the right hand side above, this determines $\beta(v_\tau)$ uniquely.

We therefore obtain a map of polynomial algebras $\mathbf{k}[v_\tau\colon \tau\in\mathcal{S}]\to\mathbf{k}[v_\tau\colon \tau\in\widetilde{\mathcal{S}}]$ (which we denote by the same letter β for a moment), and need to check that it descends to a map of face rings, $\mathbf{k}[\mathcal{S}]\to\mathbf{k}[\widetilde{\mathcal{S}}]$. In other words, we need to verify that $\beta(\mathcal{I}_{\mathcal{S}})\subset\mathcal{I}_{\widetilde{\mathcal{S}}}$.

It is clear from the definition of β that we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{k}[v_\tau\colon \tau\in\mathcal{S}] \arrow[r, "p"] \arrow[d, "\beta"] & \mathbf{k}[\mathcal{S}] \arrow[r, "s"] \arrow[d, dashed, "\beta"] & \bigoplus_{\tau\in\mathcal{S}}\mathbf{k}[v_i\colon i\in V(\tau)] \arrow[d, "s(\beta)"] \\
\mathbf{k}[v_\tau\colon \tau\in\widetilde{\mathcal{S}}] \arrow[r, "\widetilde{p}"] & \mathbf{k}[\widetilde{\mathcal{S}}] \arrow[r, "\widetilde{s}"] & \bigoplus_{\tau\in\widetilde{\mathcal{S}}}\mathbf{k}[v_i\colon i\in V(\tau)],
\end{tikzcd}$$

in which the middle vertical map is not defined yet. Here by s and $\widetilde{s}$ we denote the restriction maps from Theorem 3.5.6, and $s(\beta)$ is the map induced by β on the direct sum of polynomial algebras. Now let $x\in\mathcal{I}_{\mathcal{S}}$, i.e. $p(x)=0$. Then, by the commutativity of the diagram, $\widetilde{s}\widetilde{p}\beta(x)=0$. Since $\widetilde{s}$ is injective, we have $\widetilde{p}\beta(x)=0$. Hence, $\beta(x)\in\mathcal{I}_{\widetilde{\mathcal{S}}}$, which implies that the middle vertical map is well defined.

The rest of the statement also follows from the commutative diagram above. The map $s(\beta)$ sends each direct summand of its domain isomorphically to at least one summand of its range, and therefore it is injective. Thus, $\beta\colon\mathbf{k}[\mathcal{S}]\to\mathbf{k}[\mathcal{S}']$ is also injective. Finally, the statement about hsop's follows from the diagram and Lemma 3.5.8. $\square$

Remark. If we defined the map β by sending each v_i identically, then it would still give rise to a ring homomorphism $\mathbf{k}[\mathcal{S}]\to\mathbf{k}[\widetilde{\mathcal{S}}]$, but the latter would not be injective (for example, it would map $v_\sigma\in\mathbf{k}[\mathcal{S}]$ to zero).

Example 3.6.5. The assumption on $\mathrm{st}_{\mathcal{S}}\,\sigma$ in Lemma 3.6.4 is not always satisfied. For example, if $\mathcal{S}$ is obtained by identifying two 2-simplices along their boundaries, and σ is any edge, then $\mathrm{st}_{\mathcal{S}}\,\sigma=\mathcal{S}$, which is not a simplicial complex.

Note also that if $\mathrm{st}_{\mathcal{S}}\,\sigma$ is not a simplicial complex, then the map $\beta\colon\mathbf{k}[\mathcal{S}]\to\mathbf{k}[\mathcal{S}']$ is not determined uniquely by the conditions specified in Lemma 3.6.4 (i.e. we cannot determine the images of v_τ with $\tau\in\mathrm{st}_{\mathcal{S}}\,\sigma$). Nevertheless, it is still possible to define the map $\beta\colon\mathbf{k}[\mathcal{S}]\to\mathbf{k}[\mathcal{S}']$ for an arbitrary simplicial poset $\mathcal{S}$, see Section **??**.

Lemma 3.6.6. *Assume that $\mathbf{k}[\mathcal{S}]$ is a Cohen–Macaulay ring, and let $\widetilde{\mathcal{S}}$ be a stellar subdivision of $\mathcal{S}$ at σ such that $\mathrm{st}_{\mathcal{S}}\,\sigma$ is a simplicial complex. Then $\mathbf{k}[\widetilde{\mathcal{S}}]$ is a Cohen–Macaulay ring.*

Proof. We first prove that $\mathrm{st}_{\mathcal{S}}\,\sigma$ is a Cohen–Macaulay complex. Since $\mathrm{st}_{\mathcal{S}}\,\sigma=\Delta^\sigma*\mathrm{lk}_{\mathcal{S}}\,\sigma$, it is enough to verify that $\mathrm{lk}_{\mathcal{S}}\,\sigma$ is Cohen–Macaulay. This follows from Reisner's Theorem (Theorem 3.3.8) and the fact that simplicial cohomology of $\mathrm{lk}_{\mathcal{S}}\,\sigma$ is a direct summand in local cohomology of $\mathbf{k}[\mathcal{S}]$ (see [**184**, Th. II.4.1] or [**32**, Th. 5.3.8]).

Now choose an hsop $\boldsymbol{t}=(t_1,\dots,t_n)$ in $\mathbf{k}[\mathcal{S}]$ and set $\widetilde{\boldsymbol{t}}=\beta(\boldsymbol{t})$. Consider the projection

$$p\colon\mathbf{k}[\mathcal{S}]\to\mathbf{k}[\mathcal{S}]/(v_\tau\colon\tau\notin\mathrm{st}_{\mathcal{S}}\,\sigma)=\mathbf{k}[\mathrm{st}_{\mathcal{S}}\,\sigma]$$

and denote its kernel by R. Similarly, set

$$\widetilde{R} = \ker\bigl(\widetilde{p}\colon \mathbf{k}[\widetilde{\mathcal{S}}] \to \mathbf{k}[\operatorname{st}_{\widetilde{\mathcal{S}}} v]\bigr),$$

where v is the new vertex added in the process of stellar subdivision. Since the simplicial cell complexes $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ do not differ on the complement to $\operatorname{st}_{\mathcal{S}}\sigma$ and $\operatorname{st}_{\widetilde{\mathcal{S}}} v$ respectively, the map β restricts to the identity isomorphism $R \to \widetilde{R}$. We therefore have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "\cong"] & \mathbf{k}[\mathcal{S}] \arrow[r, "p"] \arrow[d, "\beta"] & \mathbf{k}[\operatorname{st}_{\mathcal{S}}\sigma] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{R} \arrow[r] & \mathbf{k}[\widetilde{\mathcal{S}}] \arrow[r, "\widetilde{p}"] & \mathbf{k}[\operatorname{st}_{\widetilde{\mathcal{S}}} v] \arrow[r] & 0,
\end{tikzcd}$$

Applying the functors $\otimes_{\mathbf{k}[\boldsymbol{t}]}\mathbf{k}$ and $\otimes_{\mathbf{k}[\widetilde{\boldsymbol{t}}]}\mathbf{k}$ to the diagram above, we get a map between the long exact sequences for Tor. Consider the following fragment:

$$\begin{array}{ccccccc}
\operatorname{Tor}^{-2}_{\mathbf{k}[\boldsymbol{t}]}(\mathbf{k}[\operatorname{st}\sigma],\mathbf{k}) & \xrightarrow{f} & \operatorname{Tor}^{-1}_{\mathbf{k}[\boldsymbol{t}]}(R,\mathbf{k}) & \to & \operatorname{Tor}^{-1}_{\mathbf{k}[\boldsymbol{t}]}\mathbf{k}[\mathcal{S}],\mathbf{k}) & \to & \operatorname{Tor}^{-1}_{\mathbf{k}[\boldsymbol{t}]}(\mathbf{k}[\operatorname{st}\sigma],\mathbf{k}) \\
\downarrow & & \downarrow\cong & & \downarrow & & \downarrow \\
\operatorname{Tor}^{-2}_{\mathbf{k}[\widetilde{\boldsymbol{t}}]}(\mathbf{k}[\operatorname{st}v],\mathbf{k}) & \xrightarrow{\widetilde{f}} & \operatorname{Tor}^{-1}_{\mathbf{k}[\widetilde{\boldsymbol{t}}]}(\widetilde{R},\mathbf{k}) & \to & \operatorname{Tor}^{-1}_{\mathbf{k}[\widetilde{\boldsymbol{t}}]}(\mathbf{k}[\widetilde{\mathcal{S}}],\mathbf{k}) & \to & \operatorname{Tor}^{-1}_{\mathbf{k}[\widetilde{\boldsymbol{t}}]}(\mathbf{k}[\operatorname{st}v],\mathbf{k}).
\end{array}$$

Since $\mathbf{k}[\mathcal{S}]$ is Cohen–Macaulay, $\operatorname{Tor}^{-1}_{\mathbf{k}[\boldsymbol{t}]}(\mathbf{k}[\mathcal{S}],\mathbf{k}) = 0$ and the map f is surjective. Then $\widetilde{f}$ is also surjective. Since $\operatorname{st}_{\mathcal{S}}\sigma$ is a Cohen–Macalay simplicial complex and $|\operatorname{st}_{\mathcal{S}}\sigma| \cong |\operatorname{st}_{\widetilde{\mathcal{S}}} v|$, Proposition 3.3.9 implies that $\mathbf{k}[\operatorname{st}v]$ is Cohen–Macaulay. Therefore, $\operatorname{Tor}^{-1}_{\mathbf{k}[\widetilde{\boldsymbol{t}}]}(\mathbf{k}[\operatorname{st}v],\mathbf{k}) = 0$. Since $\widetilde{f}$ is surjective, we also have $\operatorname{Tor}^{-1}_{\mathbf{k}[\widetilde{\boldsymbol{t}}]}(\mathbf{k}[\widetilde{\mathcal{S}}],\mathbf{k}) = 0$. Then $\mathbf{k}[\widetilde{\mathcal{S}}]$ is free as a $\mathbf{k}[\widetilde{\boldsymbol{t}}]$-module (see [**126**, Lemma VII.6.2]) and thereby is Cohen–Macaulay. □

Now we can prove the main result of this section:

Theorem 3.6.7. *A simplicial poset $\mathcal{S}$ is Cohen–Macaulay if and only if the face ring $\mathbf{k}[\mathcal{S}]$ is Cohen–Macaulay.*

Proof. The fact that the face ring of a Cohen–Macaulay simplicial poset $\mathcal{S}$ is Cohen–Macaulay is proved in [**182**, Cor. 3.7] (see also [**184**, § III.6]) using the theory of ASL's.

Assume now that $\mathbf{k}[\mathcal{S}]$ is a Cohen–Macaulay ring. Since the barycentric subdivision $\mathcal{S}'$ is obtained by a sequence of stellar subdivisions, subsequent application of Lemma 3.6.6 gives that $\mathbf{k}[\mathcal{S}']$ is also Cohen–Macaulay. Thus, $\mathcal{S}'$ is a Cohen–Macaulay poset. □

In the end of this section we give Stanley's characterisation of h-vectors of Cohen–Macaulay simplicial posets.

Theorem 3.6.8 (Stanley)**.** *The integer vector $\boldsymbol{h} = (h_0, h_1, \ldots, h_n)$ is the h-vector of a Cohen–Macaulay simplicial poset if and only if $h_0 = 1$ and $h_i \geqslant 0$ for any i.*

Proof. Let $\boldsymbol{h} = \boldsymbol{h}(\mathcal{S})$ for a Cohen–Macaulay simplicial poset $\mathcal{S}$. The condition $h_0 = 1$ follows from the definition of the h-vector, see (**??**). Let $\mathbf{k}$ be a field of zero characteristic, and $\boldsymbol{t} = (t_1, \ldots, t_n)$ an lsop in $\mathbf{k}[\mathcal{S}]$ (since $\mathbf{k}[\mathcal{S}]$ is not generated by linear elements, the existence of an lsop is not automatic and is left as an exercise;

alternatively, see [**182**, Lemma 3.9]). Comparing the formula for the Poincaré series from Proposition A.3.12 with that of Theorem 3.5.9, we obtain

$$F\big(\mathbf{k}[\mathcal{S}]/\boldsymbol{t};\lambda\big) = h_0 + h_1\lambda^2 + \cdots + h_n\lambda^{2n}.$$

Hence, $h_i \geqslant 0$, as needed.

Now we construct a Cohen–Macaulay simplicial cell complex $\mathcal{S}$ with any given h-vector such that $h_0 = 1$ and $h_i \geqslant 0$. First note that $\boldsymbol{h}(\Delta^{n-1}) = (1, 0, \ldots, 0)$ and Δ^{n-1} is a Cohen–Macaulay simplicial (cell) complex. Now, given an $(n-1)$-dimensional Cohen–Macaulay simplicial cell complex $\mathcal{S}$ with the h-vector $(h_0, \ldots, h_n)$, it suffices to construct, for any $k = 1, \ldots, n$, a new Cohen–Macaulay simplicial cell complex $\mathcal{S}_k$ with the h-vector given by

$$\boldsymbol{h}(\mathcal{S}_k) = (h_0, \ldots, h_{k-1}, h_k + 1, h_{k+1}, \ldots, h_n). \tag{3.15}$$

To do this, we choose an $(n-1)$-face of $\mathcal{S}$, and in this face choose some k faces of dimension $n-2$. Then add to $\mathcal{S}$ a new $(n-1)$-simplex by attaching it along some k faces of dimension $n-2$ to the chosen k faces of $\mathcal{S}$. A direct check shows that the h-vector of the resulting simplicial cell complex $\mathcal{S}_k$ is given by (3.15). The fact that $\mathcal{S}_k$ is Cohen–Macaulay follows directly from Proposition 3.3.9. □

Note that this characterisation is substantially simpler than that for simplicial complexes (see Propositions 3.3.7 and the remark after it).

3.6.1. Exercises.

Exercise 3.6.9. The map of face rings $\mathbf{k}[\mathcal{S}] \to \mathbf{k}[\widetilde{\mathcal{S}}]$ of Lemma 3.6.4 is not induced by any poset map $\widetilde{\mathcal{S}} \to \mathcal{S}$.

Exercise 3.6.10. Let $\widetilde{\mathcal{S}}$ be a stellar subdivision of $\mathcal{S}$ at σ such that $\operatorname{st}_{\mathcal{S}}\sigma$ is a simplicial complex. Show that the ring $\mathbf{k}[\mathcal{S}]$ is Cohen–Macaulay if and only if $\mathbf{k}[\widetilde{\mathcal{S}}]$ is Cohen–Macaulay, i.e. the converse of Lemma 3.6.6 holds.

Exercise 3.6.11. If $\mathbf{k}$ is of zero characteristic, then $\mathbf{k}[\mathcal{S}]$ admits an lsop.

3.7. Gorenstein simplicial posets

Gorenstein simplicial posets arise in toric topology as the combinatorial structures associated to the orbit quotients of *torus manifolds*, which are the subject of Chapter **??**. It was exactly this particular feature of Gorenstein simplicial posets which allowed Masuda [**130**] to complete the characterisation of their h-vectors, conjectured by Stanley in [**182**]. We include Masuda's result here as Theorem 3.7.4.

Definition 3.7.1. A simplicial poset $\mathcal{S}$ is *Gorenstein* (respectively, *Gorenstein**) if its barycentric subdivision $\mathcal{S}'$ is a Gorenstein (respectively, Gorenstein*) simplicial complex.

Like the Cohen–Macaulayness, the property of a simplicial poset $\mathcal{S}$ to be Gorenstein* depends only on the topology of the realisation $|\mathcal{S}|$ (this follows from Theorem 3.4.2). In particular, simplicial cell subdivisions of spheres are Gorenstein*.

The problem of characterisation of h-vectors of Gorenstein* simplicial posets complexes is more subtle than the corresponding question in the Cohen–Macaulay case. (Although this problem is much easier for simplicial posets than for simplicial complexes, see the discussion in the end of Section **??**.)

Theorem 3.7.2. *Let $\boldsymbol{h}(\mathcal{S}) = (h_0, h_1, \ldots, h_n)$ be the h-vector of a Gorenstein* simplicial poset of rank n. Then $h_0 = 1$, $h_i \geqslant 0$ and $h_i = h_{n-i}$ for any i.*

PROOF. The inequalities $h_i \geqslant 0$ follow from the fact that $\mathcal{S}$ is Cohen–Macaulay (Theorem 3.6.8). The identities $h_i = h_{n-i}$ will follow from the expression of the h-vector of the barycentric subdivision $\mathcal{S}'$ via $\boldsymbol{h}(\mathcal{S})$ and from the Dehn–Sommerville relations for the Gorenstein* simplicial complex $\mathcal{S}'$. Indeed, repeating the argument from Lemmata **??** and **??** we obtain the identity $\boldsymbol{h}(\mathcal{S}') = D\boldsymbol{h}(\mathcal{S})$, in which the vector $\boldsymbol{h}(\mathcal{S}')$ is symmetric, i.e. satisfies the Dehn–Sommerville relations. It can be checked directly using some identities for binomial coefficients that the operator D (and its inverse) takes symmetric vectors to symmetric ones (which is equivalent to the identity $d_{pq} = d_{n+1-p,n+1-q}$). This calculation can be avoided using the following argument. The Dehn–Sommerville relations specify a linear subspace W of dimension $k = \left[\frac{n}{2}\right] + 1$ in the space $\mathbb{R}^{n+1}$ with coordinates $h_0, \ldots, h_n$. We need to check that this subspace is D-invariant. To do this it suffices to choose a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ in W and check that $D\boldsymbol{e}_i \in W$ for all i. There is a basis in W consisting of h-vectors of simplicial spheres (and even simplicial polytopes, see the proof of Proposition **??**). Since the barycentric subdivision of a simplicial sphere is a simplicial sphere, the vectors $D\boldsymbol{e}_i$, $1 \leqslant i \leqslant k$, are also symmetric, and W is a D-invariant subspace. Thus, the vector $\boldsymbol{h}(\mathcal{S}) = D^{-1}\boldsymbol{h}(\mathcal{S}')$ satisfies the Dehn–Sommerville relations. □

Theorem 3.7.3 ([**182**, Th. 4.3])**.** *Let $\boldsymbol{h} = (h_0, h_1, \ldots, h_n)$ be an integer vector with $h_0 = 1$, $h_i \geqslant 0$ and $h_i = h_{n-i}$. Any of the following (mutually exclusive) conditions are sufficient for the existence of a Gorenstein* simplicial poset of rank n and h-vector $h(\mathcal{S}) = \boldsymbol{h}$:*

(a) *n is odd;*
(b) *n is even and $h_{n/2}$ is even;*
(c) *n is even, $h_{n/2}$ is odd, and $h_i > 0$ for all i.*

PROOF. We start with the following two basic examples of $(n-1)$-dimensional simplicial cell complexes of dimension: $\partial\Delta^n$, with h-vector $\boldsymbol{h}(\partial\Delta^n) = (1, 1, \ldots, 1)$; and $\mathcal{S}_n$, the simplicial cell complex obtained by identifying two $(n-1)$-simplices along their boundaries, with $\boldsymbol{h}(\mathcal{S}_n) = (1, 0, \ldots, 0, 1)$. By applying the standard operations of join and connected sum (Constructions **??** and **??**) to these two complexes we shall obtain a simplicial cell complex with any h-vector satisfying the conditions of the theorem. Indeed, for $k \neq n-k$ we have

$$\boldsymbol{h}(\mathcal{S}_k * \mathcal{S}_{n-k}) = (1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1),$$

where $h_k = h_{n-k} = 1$, and the other entries are zero. Also, for $n = 2k$ we have

$$\boldsymbol{h}(\mathcal{S}_k * \mathcal{S}_k) = (1, 0, \ldots, 0, 2, 0, \ldots, 0, 1),$$

where $h_k = 2$. Now, by taking connected sum of the appropriate number of complexes $\partial\Delta^n$, $\mathcal{S}_n$ and $\mathcal{S}_k * \mathcal{S}_{n-k}$ and using the identity

$$h_i(\mathcal{S} \mathbin{\#} \widetilde{\mathcal{S}}) = h_i(\mathcal{S}) + h_i(\widetilde{\mathcal{S}}) \quad \text{for } 1 \leqslant i \leqslant n-1,$$

(see Example **??**, which is valid for any two pure $(n-1)$-dimensional simplicial cell complexes), we obtain any required h-vector. □

The subtlest part of the characterisation of h-vectors of Gorenstein* simplicial posets is contained in the following result, which was conjectured by Stanley and proved by Masuda [**130**].

Theorem 3.7.4 (Masuda). *Let $\boldsymbol{h}(\mathcal{S})=(h_0,h_1,\dots,h_n)$ be the h-vector of a Gorenstein* simplicial poset $\mathcal{S}$ of even rank n, and let $h_i=0$ for some i. Then the number $h_{n/2}$ is even.*

Note that the evenness of $h_{n/2}$ is equivalent to the evenness of the number of facets $f_{n-1}=\sum_{i=0}^n h_i$. As we have mentioned above, the idea behind Masuda's proof of Theorem 3.7.4 lies within the topological theory of torus manifolds. We shall outline this proof in Section **??**.

We combine the results of Theorems 3.7.2, 3.7.3 and 3.7.4 in the following characterisation result for the h-vectors of Gorenstein* simplicial posets.

Theorem 3.7.5. *An integer vector $\boldsymbol{h}=(h_0,h_1,\dots,h_n)$ is the h-vector of a Gorenstein* simplicial poset of rank n if and only if the following conditions are satisfied:*

(a) *$h_0=1$ and $h_i\geqslant 0$;*
(b) *$h_i=h_{n-i}$ for all i;*
(c) *either $h_i>0$ for all i or $\sum_{i=0}^n h_i$ is even.*

The same conditions characterise also the h-vectors of simplicial cell subdivisions of spheres.

3.8. Generalised Dehn–Sommerville relations

In this section we obtain some further generalisations of the Dehn–Sommerville relations, in particular, to arbitrary triangulated manifolds.

Let $\mathcal{S}$ be a simplicial poset of rand n. Given $\sigma\in\mathcal{S}$, consider the closed upper semiinterval $\mathcal{S}_{\geqslant\sigma}=\{\tau\in\mathcal{S}\colon\tau\geqslant\sigma\}$ with the induced order relation and rank function, and set

$$\chi(\mathcal{S}_{\geqslant\sigma})=\sum_{\tau\geqslant\sigma}(-1)^{|\tau|-1}. \tag{3.16}$$

A simplicial poset $\mathcal{S}$ of rank n satisfying $\chi(\mathcal{S}_{\geqslant\sigma})=(-1)^{n-1}$ for any $\sigma\in\mathcal{S}$ is called *Eulerian*. According to a result of [**181**, (3.40)], the Dehn–Sommerville relations $h_i=h_{n-i}$ hold for Eulerian posets. This can be generalised as follows.

Theorem 3.8.1 (see [**127**, Th. 9.1]). *The following identity holds for the h-vector $\boldsymbol{h}(\mathcal{S})=(h_0,\dots,h_n)$ of a simplicial poset S of rank n:*

$$\sum_{i=0}^{n}(h_{n-i}-h_i)t^i=\sum_{\sigma\in\mathcal{S}}\Bigl(1+(-1)^n\chi(\mathcal{S}_{\geqslant\sigma})\Bigr)(t-1)^{n-|\sigma|}.$$

In particular, if $\mathcal{S}$ is Eulerian, then $h_i=h_{n-i}$.

PROOF. We have

$$
\begin{aligned}
\sum_{i=0}^{n} h_i t^i &= t^n \sum_{i=0}^{n} h_i \big(\tfrac{1}{t}\big)^{n-i} = t^n \sum_{i=0}^{n} f_{i-1} \big(\tfrac{1-t}{t}\big)^{n-i} \\
&= \sum_{i=0}^{n} f_{i-1} t^i (1-t)^{n-i} = \sum_{\tau\in\mathcal{S}} t^{|\tau|}(1-t)^{n-|\tau|} \\
&= \sum_{\tau\in\mathcal{S}} \sum_{\sigma\leqslant\tau} (t-1)^{|\tau|-|\sigma|}(1-t)^{n-|\tau|} = \sum_{\tau\in\mathcal{S}} \sum_{\sigma\leqslant\tau} (-1)^{n-|\tau|}(t-1)^{n-|\sigma|} \\
&= \sum_{\sigma\in\mathcal{S}} (t-1)^{n-|\sigma|} \sum_{\tau\geqslant\sigma} (-1)^{n-|\tau|} = \sum_{\sigma\in\mathcal{S}} (t-1)^{n-|\sigma|}(-1)^{n-1}\chi(\mathcal{S}_{\geqslant\sigma}),
\end{aligned}
\tag{3.17}
$$

where the fifth identity follows from the binomial expansion of the right hand side of the identity $t^{|\tau|} = ((t-1)+1)^{|\tau|}$ and the fact that $[\hat{0}, \tau] = \{\sigma \in \mathcal{S} \colon \sigma \leqslant \tau\}$ is a Boolean lattice of rank $|\tau|$.

On the other hand, we have

$$
\sum_{i=0}^{n} h_{n-i} t^i = \sum_{i=0}^{n} h_i t^{n-i} = \sum_{i=0}^{n} f_{i-1}(t-1)^{n-i} = \sum_{\sigma\in\mathcal{S}} (t-1)^{n-|\sigma|}. \tag{3.18}
$$

Subtracting (3.17) from (3.18) we obtain the required identity. □

As a corollary we obtain a generalisation of the Dehn–Sommerville relations to triangulated manifolds (which first appeared in [**40**, Cor. 4.5.4] as a corollary of bigraded Poincaré duality for moment-angle complexes):

Theorem 3.8.2. *Let $\mathcal{K}$ be a triangulation of a closed $(n-1)$-dimensional manifold. Then the h-vector $\boldsymbol{h}(\mathcal{K}) = (h_0, \ldots, h_n)$ satisfies the identities*

$$
h_{n-i} - h_i = (-1)^i \binom{n}{i} \big(\chi(\mathcal{K}) - \chi(S^{n-1})\big), \quad 0 \leqslant i \leqslant n.
$$

Here $\chi(\mathcal{K}) = f_0 - f_1 + \cdots + (-1)^{n-1} f_{n-1} = 1 + (-1)^{n-1} h_n$ is the Euler characteristic of $\mathcal{K}$ and $\chi(S^{n-1}) = 1 + (-1)^{n-1}$.

PROOF. Viewing $\mathcal{K}$ as a simplicial poset, we calculate

$$
\begin{aligned}
\chi(\mathcal{K}_{\geqslant\sigma}) &= \sum_{\tau>\sigma} (-1)^{|\tau|-1} + (-1)^{|\sigma|-1} = (-1)^{|\sigma|}\Big(\sum_{\tau>\sigma} (-1)^{|\tau|-|\sigma|-1} - 1\Big) \\
&= (-1)^{|\sigma|}\Big(\sum_{\varnothing\neq\rho\in \mathrm{lk}_{\mathcal{K}}\sigma} (-1)^{|\rho|-1} - 1\Big) = (-1)^{|\sigma|}\big(\chi(\mathrm{lk}_{\mathcal{K}}\sigma) - 1\big).
\end{aligned}
$$

Here we used the fact that the poset of nonempty faces of $\mathrm{lk}_{\mathcal{K}}\sigma$ is isomorphic to $\mathcal{S}_{>\sigma}$, with the rank function shifted by $|\sigma|$. Now since $\mathcal{K}$ is a triangulated $(n-1)$-dimensional manifold, the link of a nonempty face $\sigma \in \mathcal{K}$ has homology of a sphere of dimension $(n - |\sigma| - 1)$. Hence, $\chi(\mathrm{lk}_{\mathcal{K}}\sigma) = 1 + (-1)^{n-|\sigma|-1}$, and therefore $\chi(\mathcal{K}_{\geqslant\sigma}) = (-1)^{n-1}$ for $\sigma \neq \varnothing$. Also, $\mathrm{lk}_{\mathcal{K}}\varnothing = \mathcal{K}$. Now using the identity of Theorem 3.8.1 we calculate

$$
\sum_{i=0}^{n} (h_{n-i} - h_i) t^i = \big(1 + (-1)^n(\chi(\mathcal{K}) - 1)\big)(t-1)^n = (-1)^n\big(\chi(\mathcal{K}) - \chi(S^{n-1})\big)(t-1)^n.
$$

The required identity follows by comparing the coefficients of t^i. □

3.8.1. Exercises.

Exercise 3.8.3. The identity of Theorem 3.8.2 holds for arbitrary simplicial posets.

APPENDIX A

Commutative and homological algebra

Here we review some basic algebraic notions and results in a way suited for topological application. In order to make algebraic constructions compatible with topological ones we sometimes use a notation which may seem unusual to a reader of algebraic background. This in particular concerns the way we treat gradings and resolutions.

We fix a ground ring $\mathbf{k}$, which is always assumed to be a field or the ring $\mathbb{Z}$ of integers. In the latter case by a '$\mathbf{k}$-vector space' we mean an abelian group.

A.1. Algebras and modules

A $\mathbf{k}$-*algebra* (or shortly *algebra*) A is a ring which is also a $\mathbf{k}$-vector space, and whose multiplication $A\times A \to A$ is $\mathbf{k}$-bilinear. (The latter condition is void if $\mathbf{k} = \mathbb{Z}$, so $\mathbb{Z}$-algebras are ordinary rings.) All our algebras will be commutative and with unit 1, unless explicitly stated otherwise. The basic example is $A = \mathbf{k}[v_1, \dots, v_m]$, the *polynomial algebra* in m generators, for which we shall often use a shortened notation $\mathbf{k}[m]$.

An algebra A is *finitely generated* if there are finitely many elements $a_1, \dots, a_n$ of A such that every element of A can be written as a polynomial in $a_1, \dots, a_n$ with coefficients in $\mathbf{k}$. Therefore, a finitely generated algebra is the quotient of a polynomial algebra by an ideal.

An A-*module* is a $\mathbf{k}$-vector space M on which A acts linearly, that is, there is a map $A \times M \to M$ which is $\mathbf{k}$-linear in each argument and satisfies $(ab)m = a(bm)$ for all $a, b \in A$, $m \in M$. Any ideal I of A is an A-module. If $A = \mathbf{k}$, then an A-module is a $\mathbf{k}$-vector space.

An A-module M is *finitely generated* if there exist $x_1, \dots, x_n$ in M such that every element x of M can be written (not necessarily uniquely) as $x = a_1x_1 + \dots + a_nx_n$, $a_i \in A$.

An algebra A is $\mathbb{Z}$-*graded* (or shortly *graded*) if it is represented as a direct sum $A = \bigoplus_{i\in\mathbb{Z}} A^i$ such that $A^i \cdot A^j \subset A^{i+j}$. Elements $a \in A^i$ are said to be *homogeneous* of degree i, denoted $\deg a = i$. The set of homogeneous elements of A is denoted by $\mathcal{H}(A) = \bigcup_i A^i$. An ideal I of A is *homogeneous* if it is generated by homogeneous elements. In most cases our graded algebras will be either *nonpositively graded* (i.e. $A^i = 0$ for $i > 0$) or *nonnegatively graded* (i.e. $A^i = 0$ for $i < 0$); the latter is also called an $\mathbb{N}$-graded algebra. A nonnegatively graded algebra A is *connected* if $A^0 = \mathbf{k}$. For a nonnegatively graded algebra A, define the *positive ideal* by $A^+ = \bigoplus_{i>0} A^i$; if A is connected then A^+ is a maximal ideal.

If A is a graded algebra, then an A-module M is *graded* if $M = \bigoplus_{i\in\mathbb{Z}} M^i$ such that $A^i \cdot M^j \subset M^{i+j}$. An A-module map $f\colon M \to N$ between two graded modules is *degree-preserving* (or of *degree 0*) if $f(M^i) \subset N^i$, and is of *degree* k if $f(M^i) \subset N^{i+k}$ for all i.

Graded algebras arising in topology are often *graded commutative* (or *skew-commutative*) rather than commutative in the usual sense. This means that

$$ab = (-1)^{ij}\, ba \quad \text{for any } a \in A^i, b \in A^j.$$

If the characteristic of $\mathbf{k}$ is not 2, then the square of an odd-degree element in a graded commutative algebra is zero. To avoid confusion we double the grading in commutative algebras A; the resulting graded algebras $A = \bigoplus_{i\in\mathbb{Z}} A^{2i}$ are commutative in either sense.

For example, we make the polynomial algebra $\mathbf{k}[v_1, \dots, v_m]$ graded by setting $\deg v_i = 2$. It then becomes a *free* graded commutative algebra on m generators of degree two (free means no relations apart from the graded commutativity). The *exterior algebra* $\Lambda[u_1, \dots, u_m]$ has relations $u_i^2 = 0$ and $u_i u_j = -u_j u_i$. We shall assume $\deg u_i = 1$ unless otherwise specified. An exterior algebra is a free graded commutative algebra if the characteristic of $\mathbf{k}$ is not 2.

Bigraded (i.e. $\mathbb{Z} \oplus \mathbb{Z}$-graded) and *multigraded* ($\mathbb{Z}^m$-graded) algebras A are defined similarly; their homogeneous elements $a \in A$ have *bidegree* $\operatorname{bideg} a = (i, j) \in \mathbb{Z} \oplus \mathbb{Z}$ or *multidegree* $\operatorname{mdeg} a = \boldsymbol{i} \in \mathbb{Z}^m$ respectively.

A sequence of homomorphisms of A-modules

$$\cdots \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \longrightarrow \cdots$$

is called an *exact sequence* if $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$ for all i.

A *chain complex* is a sequence $C_* = \{C_i, \partial_i\}$ of A-modules C_i and homomorphisms $\partial_i \colon C_i \to C_{i-1}$ such that $\partial_i \partial_{i+1} = 0$. A chain complex is usually written as

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots$$

The condition $\partial_i \partial_{i+1} = 0$ implies that $\operatorname{Im} \partial_{i+1} \subset \operatorname{Ker} \partial_i$. The *ith homology group* (or *homology module*) of C_* is defined by

$$H_i[C_*] = \operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i+1}.$$

A *cochain complex* is a sequence $C^* = \{C^i, d^i\}$ of A-modules C^i and homomorphisms $d^i \colon C^i \to C^{i+1}$ such that $d^i d^{i-1} = 0$. A cochain complex is usually written as

$$\cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \cdots .$$

The *ith cohomology group* (or *cohomology module*) of C^* is defined by

$$H^i[C^*] = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}.$$

A cochain complex may be also viewed as a graded $\mathbf{k}$-vector space $C^* = \bigoplus_i C^i$ in which every graded component C^i is an A-module, together with an A-linear map $d \colon C^* \to C^*$ raising the degree by 1 and satisfying the condition $d^2 = 0$.

Note that a chain complex may be turned to a cochain complex by inverting the grading (i.e. turning the ith graded component into the $(-i)$th).

Let $f, g \colon C^* \to D^*$ be two maps of cochain complexes (i.e. both f and g commute with the differentials). A *cochain homotopy* between f and g is a set of maps $s = \{s^i \colon C^i \to D^{i-1}\}$ satisfying the identities

$$ds + sd = f - g$$

(more precisely, $d^{i-1}s^i + s^{i+1}d^i = f^i - g^i$). This is described by the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r, "d^{i-1}"] \arrow[dl, "s^{i-1}"'] \arrow[d] & C^i \arrow[r, "d^i"] \arrow[dl, "s^i"'] \arrow[d, "f^i-g^i"] & C^{i+1} \arrow[r] \arrow[dl, "s^{i+1}"'] \arrow[d] & \cdots \\
\cdots \arrow[r] & D^{i-1} \arrow[r, "d^{i-1}"'] & D^i \arrow[r, "d^i"'] & D^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

If there is a cochain homotopy between f and g, then f and g induce the same map in cohomology (an exercise). A chain homotopy between maps of chain complexes is defined similarly.

A *differential graded algebra* is a graded algebra A together with a $\mathbf{k}$-linear map $d\colon A \to A$, called the *differential*, which raises the degree by one, satisfies the identity $d^2 = 0$ (so that $\{A^i, d^i\}$ is a cochain complex) and the *Leibnitz identity*

$$d(a \cdot b) = da \cdot b + (-1)^i a \cdot db \quad \text{for } a \in A^i, b \in A. \tag{A.1}$$

Cohomology $H[A,d] = \operatorname{Ker} d / \operatorname{Im} d$ of a differential graded algebra A is a graded algebra (an exercise). Differential graded algebras whose differential lowers the degree by one are also considered, in which case homology is a graded algebra.

An A-module F is *free* if it is isomorphic to a direct sum $\bigoplus_{i\in I} F_i$, where each F_i is isomorphic to A as an A-module. If both A and F are graded then every F_i is isomorphic to $A^{[j]}$ for some j, where $A^{[j]}$ is the graded A-module with $(A^{[j]})^k = A^{k-j}$. A *basis* of a free A-module F is a set $\mathcal{S}$ of elements of F such that each $x \in F$ can be uniquely written as a finite linear combination of elements of $\mathcal{S}$ with coefficients in A. If A is finitely generated then all bases have the same cardinality (an exercise), called the *rank* of F. If $\mathcal{S}$ is a basis of a free A-module F, then for any A-module M a set map $\mathcal{S} \to M$ extends to an A-module homomorphism $F \to M$.

An module P is *projective* if for any epimorphism of modules $p\colon M \to N$ and homomorphism $f\colon P \to N$, there is a homomorphism $f'\colon P \to M$ such that $pf' = f$. This is described by the following commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "p"] & N \arrow[r] & 0 \\
 & P \arrow[u, "f"'] \arrow[ul, dashed, "f'"] &
\end{tikzcd}$$

Equivalently P is projective if it is a direct summand in a free module (an exercise). In particular, free modules are projective.

The *tensor product* $M \otimes_A N$ of A-modules M and N is the quotient of a free A-module on the set of generators $M \times N$ by the submodule generated by all elements of the following types:

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),$$
$$(ax, y) - a(x, y), \quad (x, ay) - a(x, y),$$

where $x, x' \in M$, $y, y' \in N$, $a \in A$. For each basis element (x, y), its image in $M \otimes_A N$ is denoted by $x \otimes y$.

We shall denote the tensor product $M \otimes_{\mathbf{k}} N$ of $\mathbf{k}$-vector spaces by simply $M \otimes N$. For example, if $M = N = \mathbf{k}[v]$, then $M \otimes N = \mathbf{k}[v_1, v_2]$.

The tensor product $A\otimes B$ of graded commutative algebras A and B is a graded commutative algebra, with the multiplication defined on homogeneous elements by

$$(a\otimes b)\cdot(a'\otimes b') = (-1)^{\deg b\deg a'}aa'\otimes bb'.$$

Exercises.

Exercise A.1.1. Cochain homotopical maps between cochain complexes induce the same maps in cohomology.

Exercise A.1.2. Cohomology of a differential graded algebra is a graded algebra.

Exercise A.1.3. If A is a finitely generated algebra, then all bases of a free A-module have the same cardinality.

Exercise A.1.4. A module is projective if and only if it is a direct summand in a free module.

A.2. Homological theory of graded rings and modules

From now we assume that A is a commutative finitely generated $\mathbf{k}$-algebra with unit, graded by nonnegative even numbers and connected. The basic example to keep in mind is $A = \mathbf{k}[m] = \mathbf{k}[v_1,\ldots,v_m]$ with $\deg v_i = 2$, however we shall need a greater generality occasionally. We also assume that all A-modules M are nonnegatively graded and finitely generated, and all module maps are degree-preserving, unless the contrary is explicitly stated.

A *free* (respectively, *projective*) *resolution* of M is an exact sequence of A-modules

$$\cdots\xrightarrow{d} R^{-i}\xrightarrow{d}\cdots\xrightarrow{d} R^{-1}\xrightarrow{d} R^0\to M\to 0 \tag{A.2}$$

in which all R^{-i} are free (respectively, projective) A-modules. A free resolution exists for every M (an exercise, or see constructions below). The minimal number p for which there exists a projective resolution (A.2) with $R^{-i}=0$ for $i>p$ is called the *projective* (or *homological*) *dimension* of the module M; we shall denote it by $\operatorname{pdim}_A M$ or shortly $\operatorname{pdim} M$. If such p does not exist, we set $\operatorname{pdim} M=\infty$. The module $M_i = \operatorname{Ker}[d\colon R^{-i+1}\to R^{-i+2}]$ is called the *ith syzygy module* for M.

Under our assumptions on A, an A-module is projective if and only if it is free (see Exercise A.2.12), and we therefore may not distinguish between free and projective resolutions.

We can convert resolution (A.2) into a bigraded differential $\mathbf{k}$-vector space $[R,d]$ with $R=\bigoplus_{i,j}R^{-i,j}$ where $R^{-i,j}=(R^{-i})^j$ is the jth graded component of the module R^{-i}, and the $(-i,j)$th component of d acts as $d^{-i,j}\colon R^{-i,j}\to R^{-i+1,j}$. We refer to the first grading of R as *exterior*; it comes from the numeration of the terms in the resolution and is therefore nonpositive by our convention. The second, *interior*, grading of R comes from the grading in the modules R^{-i} and is therefore even and nonnegative. The *total* degree of an element of R is defined as the sum of its exterior and interior degrees. We can view A as a bigraded algebra with the trivial first grading (i.e. $A^{i,j}=0$ for $i\neq 0$ and $A^{0,j}=A^j$); then R becomes a bigraded A-module.

Since $[R,d]$ is a resolution of M, the bigraded cohomology $H[R,d]$ satisfies that

$$\begin{aligned} H^{-i,j}[R,d] &= \operatorname{Ker} d^{-i,j}/\operatorname{Im} d^{-i-1,j} = 0 \quad \text{for } i>0,\\ H^{0,j}[R,d] &= M^j. \end{aligned}$$

Let $[M,0]$ be the bigraded module with zero differential and trivial exterior grading, i.e. $M^{i,j} = 0$ for $i \neq 0$ and $M^{0,j} = M^j$. Then resolution (A.2) can be interpreted as a map $[R,d] \to [M,0]$ inducing isomorphism in cohomology, i.e. a *quasi-isomorphism*. This map can be viewed as the following map of cochain complexes of A-modules:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & R^{-i} & \xrightarrow{d} & \cdots & \xrightarrow{d} & R^{-1} & \xrightarrow{d} & R^0 & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \end{array} \qquad (A.3)$$

The *Poincaré series* of a graded $\mathbf{k}$-vector space $V = \bigoplus_i V^i$ whose graded components are finite-dimensional is given by

$$F(V;\lambda) = \sum_i (\dim_{\mathbf{k}} V^i)\lambda^i.$$

Proposition A.2.1. *Let* (A.2) *be a free resolution of an A-module, in which R^{-i} is a free module of rank q_i on generators of degrees $d_{1i},\ldots,d_{q_i i}$, for $i \geqslant 0$. Then*

$$F(M;\lambda) = F(A;\lambda)\sum_{i\geqslant 0}(-1)^i(\lambda^{d_{1i}} + \cdots + \lambda^{d_{q_i i}}).$$

PROOF. Since $H^{-i,j}[R,d] = 0$ for $i > 0$ and $H^{0,j}[R,d] = M^j$, we obtain

$$\sum_{i\geqslant 0}(-1)^i \dim_{\mathbf{k}} R^{-i,j} = \dim_{\mathbf{k}} M^j$$

by the property of the Euler characteristic. Multiplying by λ^j and summing up over j we obtain

$$\sum_{i\geqslant 0}(-1)^i F(R^{-i};\lambda) = F(M;\lambda).$$

Since each of R^{-i} is a free A-module, its Poincaré series is given by $F(R^{-i};\lambda) = F(A;\lambda)(\lambda^{d_{1i}} + \cdots + \lambda^{d_{q_i i}})$, which implies the required formula. □

Construction A.2.2 (minimal resolution)**.** Let $\mathbf{k}$ be a field, and let $M = \bigoplus_{i\geqslant 0} M^i$ be a graded A-module, which is not necessarily finitely generated, but whose every graded component M^i is finite-dimensional as a $\mathbf{k}$-vector space. There is the following canonical way of constructing a free resolution for M.

Take the lowest degree i in which $M^i \neq 0$ and choose a $\mathbf{k}$-vector space basis in M^i. Span an A-submodule M_1 by this basis and then take the lowest degree in which $M \neq M_1$. In this degree choose a $\mathbf{k}$-vector space basis in the complement of M_1, and span a module M_2 by this basis and M_1. Continuing this process we obtain a system of generators for M which has a finite number of elements in each degree, and has the property that images of the generators form a basis in the $\mathbf{k}$-vector space $M \otimes_A \mathbf{k} = M/(A^+ \cdot M)$. A system of generators of M obtained in this way is referred to as *minimal* (or as a *minimal basis*).

Now choose a minimal generator set in M and span by its elements a free A-module $R^0_{\min}$. Then we have an epimorphism $R^0_{\min} \to M$. Next we choose a minimal basis in the kernel of this epimorphism, and span by it a free module $R^{-1}_{\min}$. Then choose a minimal basis in the kernel of the map $R^{-1}_{\min} \to R^0_{\min}$, and so on. On the ith step we choose a minimal basis in the kernel of the map $d\colon R^{-i+1}_{\min} \to R^{-i+2}_{\min}$ constructed on the previous step, and span a free module $R^{-i}_{\min}$ by this basis. As a

result we obtain a free resolution of M, which is referred to as *minimal*. A minimal resolution is unique up to an isomorphism.

By the construction, for any $i \geqslant 0$, the kernel of the map $d\colon R^{-i}_{\min} \to R^{-i+1}_{\min}$ is contained in $A^+ \cdot R^{-i}_{\min}$. This implies that the induced maps $R^{-i}_{\min} \otimes_A \mathbf{k} \to R^{-i+1}_{\min} \otimes_A \mathbf{k}$ are zero for $i \geqslant 1$.

Remark. If $\mathbf{k} = \mathbb{Z}$ then the above described inductive procedure still gives a minimal basis for an A-module M, but the kernel of the map $d\colon R^0_{\min} \to M$ may be not contained in $A^+ \cdot R^0_{\min}$, and the induced map $R^{-1}_{\min} \otimes_A \mathbb{Z} \to R^0_{\min} \otimes_A \mathbb{Z}$ may be nonzero.

Construction A.2.3 (Koszul resolution)**.** Let $A = \mathbf{k}[v_1, \ldots, v_m]$ and $M = \mathbf{k}$ with the A-module structure given by the augmentation map sending each v_i to zero. We turn the tensor product

$$E = E_m = \Lambda[u_1, \ldots, u_m] \otimes \mathbf{k}[v_1, \ldots, v_m]$$

into a *bigraded differential algebra* by setting

$$\begin{aligned} \operatorname{bideg} u_i &= (-1, 2), \quad \operatorname{bideg} v_i = (0, 2),\\ du_i &= v_i, \quad dv_i = 0 \end{aligned} \tag{A.4}$$

and requiring d to satisfy Leibnitz identity (A.1). Then $[E, d]$ together with the augmentation map $\varepsilon\colon E \to \mathbf{k}$ defines a cochain complex of $\mathbf{k}[m]$-modules

$$\begin{aligned} 0 \to \Lambda^m[u_1, \ldots, u_m] \otimes \mathbf{k}[v_1, \ldots, v_m] &\xrightarrow{d} \cdots \\ \xrightarrow{d} \Lambda^1[u_1, \ldots, u_m] \otimes \mathbf{k}[v_1, \ldots, v_m] &\xrightarrow{d} \mathbf{k}[v_1, \ldots, v_m] \xrightarrow{\varepsilon} \mathbf{k} \to 0, \end{aligned} \tag{A.5}$$

where $\Lambda^i[u_1, \ldots, u_m]$ is the subspace of $\Lambda[u_1, \ldots, u_m]$ generated by monomials of length i. We shall show that the complex above is an exact sequence, or equivalently, that $\varepsilon\colon [E, d] \to [\mathbf{k}, 0]$ is a quasi-isomorphism. There is an obvious inclusion $\eta\colon \mathbf{k} \to E$ such that $\varepsilon\eta = \mathrm{id}$. To finish the proof we shall construct a cochain homotopy between id and $\eta\varepsilon$, that is, a set of $\mathbf{k}$-linear maps $s = \{s^{-i,2j}\colon E^{-i,2j} \to E^{-i-1,2j}\}$ satisfying the identity

$$ds + sd = \mathrm{id} - \eta\varepsilon. \tag{A.6}$$

For $m = 1$ we define the map $s_1\colon E_1^{0,*} = \mathbf{k}[v] \to E_1^{-1,*}$ by the formula

$$s_1(a_0 + a_1 v + \cdots + a_j v^j) = (a_1 + a_2 v + \cdots + a_j v^{j-1})u.$$

Then for $f = a_0 + a_1 v + \cdots + a_j v^j \in E_1^{0,*}$ we have $ds_1 f = f - a_0 = f - \eta\varepsilon f$ and $s_1 df = 0$. On the other hand, for $fu \in E_1^{-1,*}$ we have $s_1 d(fu) = fu$ and $ds_1(fu) = 0$. In any case (A.6) holds. Now we may assume by induction that for $m = k - 1$ the required cochain homotopy $s_{k-1}\colon E_{k-1} \to E_{k-1}$ is already constructed. Since $E_k = E_{k-1} \otimes E_1$, $\varepsilon_k = \varepsilon_{k-1} \otimes \varepsilon_1$ and $\eta_k = \eta_{k-1} \otimes \eta_1$, a direct calculation shows that the map

$$s_k = s_{k-1} \otimes \mathrm{id} + \eta_{k-1}\varepsilon_{k-1} \otimes s_1$$

is a cochain homotopy between id и $\eta_k\varepsilon_k$.

Since $\Lambda^i[u_1, \ldots, u_m] \otimes \mathbf{k}[m]$ is a free $\mathbf{k}[m]$-module, (A.5) is a free resolution for the $\mathbf{k}[m]$-module $\mathbf{k}$. It is known as the *Koszul resolution*. It can be shown to be minimal (an exercise).

Let (A.2) be a projective resolution of an A-module M, and N is another A-module. Applying the functor $\otimes_A N$ to (A.3) we obtain a homomorphism of differential modules

$$[R \otimes_A N, d] \to [M \otimes_A N, 0],$$

which does not induce a cohomology isomorphism in general. The $(-i)$th graded cohomology module of the cochain complex

$$\cdots \to R^{-i} \otimes_A N \to \cdots \to R^{-1} \otimes_A N \to R^0 \otimes_A N \to 0 \tag{A.7}$$

is denoted by $\operatorname{Tor}_A^{-i}(M, N)$. We shall also consider the bigraded A-module

$$\operatorname{Tor}_A(M, N) = \bigoplus_{i,j \geqslant 0} \operatorname{Tor}_A^{-i,j}(M, N)$$

where $\operatorname{Tor}_A^{-i,j}(M, N)$ is the jth graded component of $\operatorname{Tor}_A^{-i}(M, N)$

The following properties of $\operatorname{Tor}_A^{-i}(M, N)$ are well-known (see e.g. [**126**]).

Proposition A.2.4. (a) *The module* $\operatorname{Tor}_A^{-i}(M, N)$ *does not depend, up to isomorphism, on a choice of resolution* (A.2)*;*

(b) $\operatorname{Tor}_A^{-i}(\,\cdot\,, N)$ *and* $\operatorname{Tor}_A^{-i}(M, \,\cdot\,)$ *are covariant functors;*

(c) $\operatorname{Tor}_A^{0}(M, N) = M \otimes_A N$*;*

(d) $\operatorname{Tor}_A^{-i}(M, N) \cong \operatorname{Tor}_A^{-i}(N, M)$*;*

(e) *An exact sequence of A-modules*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

induces the following long exact sequence:

$$\begin{aligned}
\cdots \longrightarrow \operatorname{Tor}_A^{-i}(M_1, N) \longrightarrow \operatorname{Tor}_A^{-i}(M_2, N) \longrightarrow \operatorname{Tor}_A^{-i}(M_3, N) \longrightarrow \cdots \\
\cdots \longrightarrow \operatorname{Tor}_A^{-1}(M_1, N) \longrightarrow \operatorname{Tor}_A^{-1}(M_2, N) \longrightarrow \operatorname{Tor}_A^{-1}(M_3, N) \\
\longrightarrow \operatorname{Tor}_A^{0}(M_1, N) \longrightarrow \operatorname{Tor}_A^{0}(M_2, N) \longrightarrow \operatorname{Tor}_A^{0}(M_3, N) \longrightarrow 0.
\end{aligned}$$

In the case $N = \mathbf{k}$ the Tor-modules can be read from a minimal resolution of M as follows:

Proposition A.2.5. *Let* $\mathbf{k}$ *be a field, and let* (A.2) *be a minimal resolution of an A-module M. Then*

$$\operatorname{Tor}_A^{-i}(M, \mathbf{k}) \cong R_{\min}^{-i} \otimes_A \mathbf{k},$$
$$\dim_{\mathbf{k}} \operatorname{Tor}_A^{-i}(M, \mathbf{k}) = \operatorname{rank} R_{\min}^{-i}.$$

PROOF. Indeed, the differentials in the cochain complex

$$\cdots \longrightarrow R_{\min}^{-i} \otimes_A \mathbf{k} \longrightarrow \cdots \longrightarrow R_{\min}^{-1} \otimes_A \mathbf{k} \longrightarrow R_{\min}^{0} \otimes_A \mathbf{k} \longrightarrow 0$$

are all trivial by the definition of a minimal resolution. □

Corollary A.2.6. *Let* $\mathbf{k}$ *be a field, and let M be a A-module. Then*

$$\operatorname{pdim} M = \max\bigl\{i\colon\ \operatorname{Tor}_A^{-i}(M, \mathbf{k}) \neq 0\bigr\}.$$

Corollary A.2.7. *If* $\mathbf{k}$ *is a field, then* $\operatorname{pdim} M \leqslant m$ *for any* $\mathbf{k}[v_1, \ldots, v_m]$*-module M.*

Proof. By the previous corollary and Proposition A.2.4 (d),

$$\operatorname{pdim} M = \max\bigl\{i\colon \operatorname{Tor}^{-i}_{\mathbf{k}[m]}(M,\mathbf{k}) \neq 0\bigr\} = \max\bigl\{i\colon \operatorname{Tor}^{-i}_{\mathbf{k}[m]}(\mathbf{k},M) \neq 0\bigr\}$$

Using the Koszul resolution for the $\mathbf{k}[m]$-module $\mathbf{k}$ we obtain

$$\operatorname{Tor}^{-i}_{\mathbf{k}[m]}(\mathbf{k},M) = H^{-i}\bigl[\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[m]\otimes_{\mathbf{k}[m]}M,d\bigr] = H^{-i}\bigl[\Lambda[u_1,\dots,u_m]\otimes M,d\bigr].$$

Therefore,

$$\operatorname{pdim} M = \max\bigl\{i\colon \operatorname{Tor}^{-i}_{\mathbf{k}[m]}(\mathbf{k},M) \neq 0\bigr\} \leqslant \max\bigl\{i\colon \Lambda^i[u_1,\dots,u_m]\otimes M \neq 0\bigr\} = m.$$

□

Example A.2.8. Let $A = \mathbf{k}[v_1,\dots,v_m]$ and $M = N = \mathbf{k}$. By the minimality of the Koszul resolution,

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k},\mathbf{k}) = \Lambda[u_1,\dots,u_m]$$

and $\operatorname{pdim}_A \mathbf{k} = m$.

We note that in general there is no canonical way to define a multiplication in $\operatorname{Tor}_A(M,N)$, even if both M and N are A-algebras rather than just A-modules. However, in the particular case when $A = \mathbf{k}[m]$, M is an algebra with a unit and $N = \mathbf{k}$ there is the following canonical way to define a product in $\operatorname{Tor}_A(M,N)$, extending the previous example. We consider the differential bigraded algebra $[\Lambda[u_1,\dots,u_m]\otimes M,d]$ whose bigrading and differential are defined similarly to (A.4):

$$\begin{gathered}\operatorname{bideg} u_i = (-1,2),\quad \operatorname{bideg} x = (0,\deg x)\quad \text{for } x\in M,\\ du_i = v_i\cdot 1,\quad dx = 0\end{gathered}\tag{A.8}$$

(here $v_i\cdot 1$ is the element of M obtained by applying $v_i\in\mathbf{k}[m]$ to $1\in M$, and we identify u_i with $u_i\otimes 1$ and x with $1\otimes x$ for simplicity). Using the fact that cohomology of a differential graded algebra is a graded algebra we obtain:

Lemma A.2.9. *Let M be a graded $\mathbf{k}[v_1,\dots,v_m]$-algebra. Then $\operatorname{Tor}_{\mathbf{k}[m]}(M,\mathbf{k})$ is a bigraded $\mathbf{k}$-algebra whose product is defined via the isomorphism*

$$\operatorname{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(M,\mathbf{k}\bigr) \cong H\bigl[\Lambda[u_1,\dots,u_m]\otimes M,d\bigr].$$

Proof. Using the Koszul resolution in the definition of $\operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k},M)$ and Proposition A.2.4 (d) we calculate

$$\begin{aligned}\operatorname{Tor}_{\mathbf{k}[m]}(M,\mathbf{k}) &\cong \operatorname{Tor}_{\mathbf{k}[m]}(\mathbf{k},M)\\ &= H\bigl[\Lambda[u_1,\dots,u_m]\otimes\mathbf{k}[m]\otimes_{\mathbf{k}[m]}M,d\bigr] \cong H\bigl[\Lambda[u_1,\dots,u_m]\otimes M,d\bigr].\end{aligned}$$

□

The algebra $\bigl[\Lambda[u_1,\dots,u_m]\otimes M,d\bigr]$ is known as the *Koszul algebra* (or *Koszul complex*) of M.

Lemma A.2.9 holds also in the case when M does not have unit (e.g., it is a graded ideal in $\mathbf{k}[m]$). In this case formula (A.8) for the differential needs to be updated as follows:

$$d(u_i x) = v_i\cdot x,\quad dx = 0\quad \text{for } x\in M.$$

If A is not necessarily commutative, then $\operatorname{Tor}_A(M,N)$ is defined for a right A-module M and a left A-module N in the same way as above. However, in this case $\operatorname{Tor}_A(M,N)$ is no longer an A-module, and is just a $\mathbf{k}$-vector space. If both M and N are A-bimodules, then $\operatorname{Tor}_A(M,N)$ is an A-bimodule itself.

The construction of Tor can be also extended to the case of differential graded modules and algebras, see Section A.4.

In the standard notation adopted in the algebraic literature, the modules in a resolution (A.2) are numbered by nonnegative rather than nonpositive integers:

$$\cdots \xrightarrow{d} R^i \xrightarrow{d} \cdots \xrightarrow{d} R^1 \xrightarrow{d} R^0 \to M \to 0.$$

In this notation, the ith Tor-module is denoted by $\mathrm{Tor}_i^A(M,N)$, $\quad i \geqslant 0$ (note that (A.7) becomes a chain complex, and $\mathrm{Tor}_*^A(M,N)$ is its homology). Therefore, the two notations are related by

$$\mathrm{Tor}_A^{-i}(M,N) = \mathrm{Tor}_i^A(M,N).$$

Applying the functor $\mathrm{Hom}_A(\quad, N)$ to (A.3) (with R^{-i} replaced by R^i) we obtain the cochain complex

$$0 \to \mathrm{Hom}_A(R^0,N) \to \mathrm{Hom}_A(R^1,N) \to \cdots \to \mathrm{Hom}_A(R^i,N) \to \cdots .$$

Its ith cohomology module is denoted by $\mathrm{Ext}_A^i(M,N)$.

The properties of the functor Ext are similar to those given by Proposition A.2.4 for Tor, with the exception of (d):

Proposition A.2.10. (a) *The module* $\mathrm{Ext}_A^i(M,N)$ *does not depend, up to isomorphism, on a choice of resolution* (A.2)*;*

(b) $\mathrm{Ext}_A^i(\,\cdot\,,N)$ *is a contravariant functor, and* $\mathrm{Ext}_A^i(M,\,\cdot\,)$ *is a covariant functor;*

(c) $\mathrm{Ext}_A^0(M,N) = \mathrm{Hom}_A(M,N)$*;*

(d) *An exact sequence of A-modules*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

induces the following long exact sequence:

$$\begin{aligned}
0 &\longrightarrow \mathrm{Ext}_A^0(M_3,N) \longrightarrow \mathrm{Ext}_A^0(M_2,N) \longrightarrow \mathrm{Ext}_A^0(M_1,N)\\
&\longrightarrow \mathrm{Ext}_A^1(M_3,N) \longrightarrow \mathrm{Ext}_A^1(M_2,N) \longrightarrow \mathrm{Ext}_A^1(M_1,N) \longrightarrow \cdots\\
\cdots &\longrightarrow \mathrm{Ext}_A^i(M_3,N) \longrightarrow \mathrm{Ext}_A^i(M_2,N) \longrightarrow \mathrm{Ext}_A^i(M_1,N) \longrightarrow \cdots ;
\end{aligned}$$

(e) *An exact sequence of A-modules*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

induces the following long exact sequence:

$$\begin{aligned}
0 &\longrightarrow \mathrm{Ext}_A^0(M,N_1) \longrightarrow \mathrm{Ext}_A^0(M,N_2) \longrightarrow \mathrm{Ext}_A^0(M,N_3)\\
&\longrightarrow \mathrm{Ext}_A^1(M,N_1) \longrightarrow \mathrm{Ext}_A^1(M,N_2) \longrightarrow \mathrm{Ext}_A^1(M,N_3) \longrightarrow \cdots\\
\cdots &\longrightarrow \mathrm{Ext}_A^i(M,N_1) \longrightarrow \mathrm{Ext}_A^i(M,N_2) \longrightarrow \mathrm{Ext}_A^i(M,N_3) \longrightarrow \cdots .
\end{aligned}$$

Exercises.

Exercise A.2.11. Show that a free resolution exists for every A-module M. (Hint: use the fact that every module is the quotient of a free module.)

Exercise A.2.12. If $\mathbf{k}$ is a field and $A = \mathbf{k}[m]$, then every projective graded A-module is free (hint: see [**126**, Lemma VII.6.2]). This is also true in the ungraded case, but is much harder to prove (a theorem of Quillen and Suslin, settling the famous problem of Serre). More generally, if A is a finitely generated nonnegatively

graded commutative connected algebra over a field $\mathbf{k}$, then every projective A-module is free (see [**78**, Theorem A3.2]). Give an example of a projective module over a ring which is not free.

Exercise A.2.13. The Koszul resolution is minimal.

A.3. Regular sequences and Cohen–Macaulay algebras

Cohen–Macaulay algebras and modules play an important role in commutative algebra, algebraic geometry and combinatorics. Their definition uses the notion of a regular sequence (see Definition A.3.1 below), which also played an important role in algebraic topology, namely in the construction of new cohomology theories (see [**122**] and Appendix Section **??**). In the case of finitely generated algebras over a field $\mathbf{k}$, an algebra is Cohen–Macaulay if and only if it is a free module of finite rank over its polynomial subalgebra.

Here we consider nonnegatively evenly graded finitely generated commutative connected algebras A over a field $\mathbf{k}$ and finitely generated nonnegatively graded A-modules M (the case $\mathbf{k} = \mathbb{Z}$ requires extra care, and is treated separately in some particular cases in the main chapters of the book). The positive part A^+ is a unique homogeneous maximal ideal of A, and the results we discuss here are parallel to those from the homological theory of Nötherian local rings (we refer to [**32**, Ch. 1-2] or [**78**, Ch. 19] for the details).

Given a sequence of elements $\boldsymbol{t} = (t_1, \ldots, t_k)$ of A, we denote by $A/\boldsymbol{t}$ the quotient algebra of A by the ideal generated by $\boldsymbol{t}$, and denote by $M/\boldsymbol{t}M$ the quotient module of M by the submodule $t_1M + \cdots + t_kM$. An element $t \in A$ is called a *zero divisor* on M if $tx = 0$ for some nonzero $x \in M$. An element $t \in A$ is not a zero divisor on M if and only if the map $M \xrightarrow{t} M$ given by multiplication by t is injective.

Definition A.3.1. Let M be an A-module. A homogeneous sequence $\boldsymbol{t} = (t_1, \ldots, t_k) \in \mathcal{H}(A^+)$ is called an *M-regular sequence* if t_{i+1} is not a zero divisor on $M/(t_1M + \cdots + t_iM)$ for $0 \leqslant i < k$. We often refer to A-regular sequences as *regular*.

The importance of regular sequences in homological algebra builds on the fundamental fact that an exact sequence of modules remains exact after taking quotients by a regular sequence:

Proposition A.3.2. *Assume given an exact sequence of A-modules:*

$$\cdots \longrightarrow S^i \xrightarrow{f_i} S^{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} S^0 \xrightarrow{f_0} M \to 0$$

If $\boldsymbol{t}$ is an M-regular and S^i-regular sequence for any $i \geqslant 0$, then the sequence of $A/\boldsymbol{t}$-modules

$$\cdots \longrightarrow S^i/\boldsymbol{t}S^i \xrightarrow{\overline{f}_i} S^{i-1}/\boldsymbol{t}S^{i-1} \xrightarrow{\overline{f}_{i-1}} \cdots \xrightarrow{\overline{f}_1} S^0/\boldsymbol{t}S^0 \xrightarrow{\overline{f}_0} M/\boldsymbol{t}M \to 0$$

is also exact.

Proof. Using induction we reduce the statement to the case when $\boldsymbol{t}$ consists of a single element t. Since

$$S^i/tS^i = S^i \otimes_A (A/t),$$

and $\otimes_A (A/t)$ is a right exact functor, it is enough to verify exactness of the quotient sequence starting from the term S^1/tS^1.

Consider the following fragment of the quotient sequence ($i \geqslant 1$):

$$S^{i+1}/tS^{i+1} \xrightarrow{\overline{f}_{i+1}} S^i/tS^i \xrightarrow{\overline{f}_i} S^{i-1}/tS^{i-1} \xrightarrow{\overline{f}_{i-1}} S^{i-2}/tS^{i-2}$$

(where we denote $S^{-1} = M$). For any element $x \in S^i$ we denote by $\overline{x}$ its residue class in S^i/tS^i. Let $\overline{f}_i(\overline{x}) = 0$, then $f_i(x) = ty$ for some $y \in S^{i-1}$ and $tf_{i-1}(y) = 0$. Since t is S^{i-2}-regular, we have $f_{i-1}(y) = 0$. Hence, there is $x' \in S^i$ such that $y = f_i(x')$. This implies that $f_i(x - tx') = 0$. Therefore, $x - tx' \in f_{i+1}(S^{i+1})$ and $\overline{x} \in \overline{f}_{i+1}(S^{i+1}/tS^{i+1})$. Thus, the quotient sequence is exact. □

Corollary A.3.3. *Let $\boldsymbol{t}$ be a sequence of elements of A which is A-regular and M-regular. Then*

$$\mathrm{Tor}_A(M, \mathbf{k}) = \mathrm{Tor}_{A/\boldsymbol{t}}(M/\boldsymbol{t}M, \mathbf{k}).$$

Proof. Applying Proposition A.3.2 to a minimal resolution of M, we obtain a minimal resolution of the $A/\boldsymbol{t}$-module $M/\boldsymbol{t}M$. The rest follows from Proposition A.2.5. □

An M-regular sequence is *maximal* if it is not contained in an M-regular sequence of a greater length.

Theorem A.3.4 (D. Rees)**.** *All maximal regular M-sequences in A have the same length given by*

$$\operatorname{depth}_A M = \min\{i\colon\ \mathrm{Ext}^i_A(\mathbf{k}, M) \neq 0\}. \qquad \text{(A.9)}$$

This number given by (A.9) is referred to as the *depth* of M; the simplified notation $\operatorname{depth} M$ will be used whenever it creates no confusion. The proof of Theorem A.3.4 uses the following fact:

Lemma A.3.5. *Let $\boldsymbol{t} = (t_1, \ldots, t_n) \in \mathcal{H}(A^+)$ be a regular M-sequence. Then*

$$\mathrm{Ext}^n_A(k, M) \cong \mathrm{Hom}_A(k, M/\boldsymbol{t}M).$$

Proof. We use induction on n. The case $n = 0$ is tautological. Since t_n is an M-regular element, we have the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot t_n} M \longrightarrow M/t_nM \longrightarrow 0.$$

The map $\mathrm{Ext}^i_A(\mathbf{k}, M) \to \mathrm{Ext}^i_A(\mathbf{k}, M)$ induced by the multiplication by t_n is zero (an exercise). Therefore, the second long exact sequence for Ext (see Proposition A.2.10 (e)) induced by the short exact sequence above splits into short exact sequences of the form

$$0 \longrightarrow \mathrm{Ext}^{n-1}_A(\mathbf{k}, M) \longrightarrow \mathrm{Ext}^{n-1}_A(\mathbf{k}, M/t_nM) \longrightarrow \mathrm{Ext}^n_A(\mathbf{k}, M) \longrightarrow 0.$$

Let $\boldsymbol{t}' = (t_1, \ldots, t_{n-1})$. By induction,

$$\mathrm{Ext}^{n-1}_A(\mathbf{k}, M) \cong \mathrm{Hom}_A(\mathbf{k}, M/\boldsymbol{t}'M) = 0,$$

where the latter identity follows from Exercise A.3.14, since t_n is $M/\boldsymbol{t}'M$-regular. Now the exact sequence above implies that

$$\mathrm{Ext}^n_A(\mathbf{k}, M) \cong \mathrm{Ext}^{n-1}_A(\mathbf{k}, M/t_nM) \cong \mathrm{Hom}_A(k, M/\boldsymbol{t}M),$$

where the latter identity follows by induction. □

Proof of Theorem A.3.4. Let $\boldsymbol{t} = (t_1, \dots, t_n)$ be a maximal regular M-sequence. Then, by Lemma A.3.5 and Exercise A.3.14,

$$\operatorname{Ext}^n_A(\mathbf{k}, M) \cong \operatorname{Hom}_A(\mathbf{k}, M/\boldsymbol{t}M) \neq 0,$$

as A does not contain an $M/\boldsymbol{t}M$-regular element. On the other hand,

$$\operatorname{Ext}^i_A(\mathbf{k}, M) \cong \operatorname{Hom}_A\bigl(\mathbf{k}, M/(t_1M + \dots + t_iM)\bigr) = 0$$

for $i < n$, since t_{i+1} is $M/(t_1M + \dots + t_iM)$-regular. □

The following fundamental result relates the depth to the projective dimension of a module.

Theorem A.3.6 (Auslander–Buchsbaum)**.** *Let $M \neq 0$ be an A-module such that $\operatorname{pdim} M < \infty$. Then*

$$\operatorname{pdim} M + \operatorname{depth} M = \operatorname{depth} A.$$

Proof. First let $\operatorname{depth} A = 0$. Assume that $\operatorname{pdim} M = p > 0$. Consider the minimal resolution for M (which is finite by the assumption):

$$0 \to R^{-p}_{\min} \xrightarrow{d_p} R^{-p+1}_{\min} \longrightarrow \cdots \longrightarrow R^0_{\min} \longrightarrow M \to 0\cdot$$

Since $\operatorname{depth} A = 0$, we have $\operatorname{Hom}_A(\mathbf{k}, A) = \operatorname{Ext}^0_A(\mathbf{k}, A) \neq 0$ by Theorem A.3.4. Therefore, there is a monomorphism of A-modules $i\colon \mathbf{k} \to A$. In the commutative diagram

$$\begin{array}{ccc} R^{-p} \otimes_A \mathbf{k} & \xrightarrow{d_p \otimes_A \mathbf{k}} & R^{-p+1} \otimes_A \mathbf{k} \\ {\scriptstyle \mathrm{id}\otimes_A i}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes_A i} \\ R^{-p} & \xrightarrow{d_p} & R^{-p+1} \end{array}$$

the maps d_p and $\mathrm{id} \otimes_A i$ are injective (the latter because the module R^{-p} is free). Hence, $d_p \otimes_A \mathbf{k}$ is also injective, which contradicts minimality of the resolution. Hence, $\operatorname{pdim} M = 0$, which implies that M is a free A-module and $\operatorname{depth} M = \operatorname{depth} A = 0$.

Now let $\operatorname{depth} A > 0$. Assume that $\operatorname{depth} M = 0$. Consider the first syzygy module $M_1 = \operatorname{Ker}[R^0 \to M]$ for M. It follows from (A.9) and the exact sequence for Ext that $\operatorname{depth} M_1 = 1$. Since $\operatorname{pdim} M_1 = \operatorname{pdim} M - 1$, it is enough to prove the Auslender–Buchsbaum formula for the module M_1. Hence, we may assume that $\operatorname{depth} M > 0$. This implies that there is an element $t \in A$ which is A-regular and M-regular (an exercise). Then

$$\operatorname{depth}_{A/t} A/t = \operatorname{depth}_A A - 1, \quad \operatorname{depth}_{A/t} M/tM = \operatorname{depth}_A M - 1$$

by the definition of depth, and

$$\operatorname{pdim}_{A/t} M/tM = \operatorname{pdim}_A M$$

by Proposition A.2.6 and Corollary A.3.3. Now the proof is finished by induction on $\operatorname{depth} A$. □

The (Krull) *dimension* of A, denoted $\dim A$, is the maximal number of (homogeneous) elements of A algebraically independent over $\mathbf{k}$. The *dimension* of an A-module M is defined as $\dim M = \dim(A/\operatorname{Ann} M)$, where

$$\operatorname{Ann} M = \{a \in A \colon aM = 0\}$$

is the *annihilator* of M.

Definition A.3.7. A sequence $t_1,\dots,t_n$ of algebraically independent homogeneous elements of A is called a *homogeneous system of parameters* (shortly *hsop*) for M if $\dim M/(t_1M+\dots+t_nM)=0$. Equivalently, $t_1,\dots,t_n$ is an hsop if $n=\dim M$ and M is a finitely-generated $\mathbf{k}[t_1,\dots,t_n]$-module.

The following result (due to Hilbert) is a graded version of the well-known *Nöther normalisation lemma*:

Theorem A.3.8 ([**32**, Th. 1.5.17])**.** *An hsop exists for any A-module M. If $\mathbf{k}$ is of zero characteristic and A is generated by degree-two (i.e. linear) elements, then a degree-two hsop can be chosen for M.*

An hsop consisting of linear elements is referred to as a *linear system of parameters* (shortly *lsop*).

It is easy to see that a regular sequence consists of algebraically independent elements, which implies that $\operatorname{depth} M \leqslant \dim M$.

Definition A.3.9. M is a *Cohen–Macaulay A-module* if $\operatorname{depth} M=\dim M$, that is, if A contains an M-regular sequence $t_1,\dots,t_n$ of length $n=\dim M$. If A is a Cohen–Macaulay A-module, then it is called a *Cohen–Macaulay algebra*.

The following proposition provides an alternative definition of Cohen–Macaulay algebras and modules.

Proposition A.3.10. *A sequence $t_1,\dots,t_k\in\mathcal{H}(A^+)$ is M-regular if and only if M is a free (not necessarily finitely generated) $\mathbf{k}[t_1,\dots,t_k]$-module. In particular, A is a Cohen–Macaulay algebra if and only if it is a free finitely generated module over its polynomial subalgebra.*

Proof. If M is a free $\mathbf{k}[t_1,\dots,t_k]$-module, then $M/(t_1M+\dots+t_{i-1}M)$ is a free $\mathbf{k}[t_i,\dots,t_k]$-module, which implies that t_i is $M/(t_1M+\dots+t_{i-1}M)$-regular for $1\leqslant i\leqslant k$. Therefore, $t_1,\dots,t_k$ is an M-regular sequence.

Conversely, let $\boldsymbol{t}=(t_1,\dots,t_k)$ be an M-regular sequence. Consider a minimal resolution $[R_{\min},d]$ for the $\mathbf{k}[\boldsymbol{t}]$-module M. Then, by Proposition A.3.2, the sequence of $\mathbf{k}$-modules

$$\cdots\longrightarrow R^{-1}_{\min}/\boldsymbol{t}R^{-1}_{\min}\longrightarrow R^{0}_{\min}/\boldsymbol{t}R^{0}_{\min}\longrightarrow M/\boldsymbol{t}\longrightarrow 0$$

is exact. Note that $R^{-i}_{\min}/\boldsymbol{t}R^{-i}_{\min}=R^{-i}_{\min}\otimes_{\mathbf{k}[\boldsymbol{t}]}\mathbf{k}$. Since the resolution is minimal, the map $R^{0}_{\min}\otimes_{\mathbf{k}[\boldsymbol{t}]}\mathbf{k}\to M\otimes_{\mathbf{k}[\boldsymbol{t}]}\mathbf{k}$ is an isomorphism. Hence, $R^{-i}_{\min}\otimes_{\mathbf{k}[\boldsymbol{t}]}\mathbf{k}=0$ for $i>0$, which implies that $R^{-i}_{\min}=0$. Thus, $R^{0}_{\min}\to M$ is an isomorphism, i.e. M is a free $\mathbf{k}[\boldsymbol{t}]$-module. $\square$

Proposition A.3.10 also implies that the property of being a regular sequence does not depend on the order of elements in $\boldsymbol{t}$.

Proposition A.3.11. *Let M be a Cohen–Macaulay A-module. Then a sequence $\boldsymbol{t}=(t_1,\dots,t_k)\in\mathcal{H}(A^+)$ is M-regular if and only if it is a part of an hsop for M.*

Proof. Let $\dim M=n$. Assume that $\boldsymbol{t}$ is an M-regular sequence. The fact that t_i is an $M/(t_1M+\dots+t_{i-1}M)$-regular element implies that

$$\dim A/(t_1,\dots,t_i)=\dim A/(t_1,\dots,t_{i-1})-1$$

for $i=1,\dots,k$ (an exercise). Therefore, $\dim A/\boldsymbol{t}=n-k$, i.e. $\boldsymbol{t}$ is a part of an hsop for M.

For the other direction, see [**32**, 2.1.2]. $\square$

In particular, any hsop in a Cohen–Macaulay algebra A is regular.

Proposition A.3.12. *If A is Cohen–Macaulay with an lsop $\boldsymbol{t} = (t_1, \dots, t_n)$, then there is the following formula for the Poincaré series of A:*

$$F(A; \lambda) = \frac{F\bigl(A/(t_1, \dots, t_n); \lambda\bigr)}{(1 - \lambda^2)^n},$$

where $F(A/(t_1, \dots, t_n); \lambda)$ is a polynomial with nonnegative integer coefficients.

Proof. Since A is a free finitely generated module over $\mathbf{k}[t_1, \dots, t_n]$, we have an isomorphism of $\mathbf{k}$-vector spaces $A \cong (A/\boldsymbol{t}) \otimes \mathbf{k}[t_1, \dots, t_n]$. Calculaing the Poincaré series of both sides we obtain the required formula. $\square$

Remark. If A is generated by its elements $a_1, \dots, a_n$ of positive degrees $d_1, \dots, d_n$ respectively, then it may be shown that the Poincaré series of A is a rational function of the form

$$F(A; \lambda) = \frac{P(\lambda)}{(1 - \lambda^{d_1})(1 - \lambda^{d_2}) \cdots (1 - \lambda^{d_n})},$$

where $P(\lambda)$ is a polynomial with integer coefficients. However, in general the polynomial $P(\lambda)$ cannot be given explicitly, and some of its coefficients may be negative.

Exercises.

Exercise A.3.13. The map $\mathrm{Ext}^i_A(\mathbf{k}, M) \to \mathrm{Ext}^i_A(\mathbf{k}, M)$ induced by the multiplication by an element $x \in \mathcal{H}(A^+)$ is zero.

Exercise A.3.14. The following conditions are equivalent for an A-module M:

(a) Every element of $\mathcal{H}(A^+)$ is a zero divisor on M, i.e. $\operatorname{depth} M = 0$;
(b) $\mathrm{Hom}_A(\mathbf{k}, M) \neq 0$.

(Hint: show that if $\mathcal{H}(A^+)$ consists of zero divisors on M then the ideal A^+ annihilates a homogeneous element of M, see [**78**, Cor. 3.2].)

Exercise A.3.15. Let $\operatorname{depth} A > 0$, let M be an A-module with $\operatorname{depth} M = 0$, and let $M_1 = \mathrm{Ker}[R^0 \to M]$ be the first syzygy module for M. Then $\operatorname{depth} M_1 = 1$.

Exercise A.3.16. If $\operatorname{depth} A > 0$ and $\operatorname{depth} M > 0$, then there exists an element $t \in A$ which is A-regular and M-regular.

Exercise A.3.17. The Auslander–Buchsbaum formula (Theorem A.3.6) does not hold if $\operatorname{pdim} M = \infty$.

Exercise A.3.18. Show that $\dim A = 0$ if and only if A is finite-dimensional as a $\mathbf{k}$-vector space. Is it true that $\operatorname{depth} A = 0$ implies that $\dim_{\mathbf{k}} A$ is finite?

Exercise A.3.19. Give an example of an algebra A over a field $\mathbf{k}$ of finite characteristic which is generated by linear elements, but does not have an lsop.

Exercise A.3.20. A regular sequence consists of algebraically independent elements.

Exercise A.3.21. If $t \in \mathcal{H}(A^+)$ is an M-regular element, then $\dim M/tM = \dim M - 1$.

Exercise A.3.22. Let $\mathbf{k} = \mathbb{Z}$. Show that if A is a free finitely generated module over a polynomial subalgebra $\mathbb{Z}[t_1, \ldots, t_k]$ then t_{i+1} is not a zero divisor on $A/(t_1, \ldots, t_i)$ for $0 \leqslant i < k$, but the converse is not true. Therefore, the two possible definitions of a regular sequence over $\mathbb{Z}$ do not agree. (The reason why Proposition A.3.10 fails over $\mathbb{Z}$ is that minimal resolutions do not have required good properties, see the remark after Construction A.2.2.)

A.4. Eilenberg–Moore spectral sequences

In their paper [**77**] of 1966, Eilenberg and Moore constructed a spectral sequence, which became one of the important calculation tools of algebraic topology. It particular, it provides a method for calculation of cohomology of the fibre of a bundle $E \to B$ using the canonical $H^*(B)$-module structure in $H^*(E)$. This spectral sequence can be considered as an extension of Adams' approach to calculating cohomology of loop spaces [**1**]. In the 1960–70s applications of the Eilenberg–Moore spectral sequence led to many important results on cohomology of homogeneous spaces for Lie groups. More recently it has been used for different calculations with toric spaces. This appendix section contains the necessary information about the spectral sequence; we mainly follow L. Smith's paper [**177**] in this description. For a detailed account of differential homological algebra and the Eilenberg–Moore spectral sequence, as well as its applications which go beyond the scope of this book, we refer to McCleary's book [**134**].

Here we assume that $\mathbf{k}$ is a field. The following theorem provides an algebraic setup for the Eilenberg–Moore spectral sequence.

Theorem A.4.1 (Eilenberg–Moore [**177**, Theorem 1.2])**.** *Let A be a differential graded $\mathbf{k}$-algebra, and let M, N be differential graded A-modules. Then there exists a spectral sequence $\{E_r, d_r\}$ converging to $\operatorname{Tor}_A(M,N)$ and whose E_2-term is*

$$E_2^{-i,j} = \operatorname{Tor}^{-i,j}_{H[A]}\bigl(H[M], H[N]\bigr), \quad i, j \geqslant 0,$$

where $H[\cdot]$ denotes the algebra or module of cohomology.

Remark. The construction of Tor for differential graded objects requires some additional considerations (see e.g. [**177**] or [**126**, Chapter XII]).

The spectral sequence of Theorem A.4.1 lives in the second quadrant and its differentials d_r add $(r, 1-r)$ to the bidegree, for $r \geqslant 1$. We shall refer to it as the *algebraic Eilenberg–Moore spectral sequence.* Its E_∞-term is expressed via a certain decreasing filtration $\{F^{-p}\operatorname{Tor}_A(M,N)\}$ in $\operatorname{Tor}_A(M,N)$ by the formula

$$E_\infty^{-p,n+p} = F^{-p}\Bigl(\sum_{-i+j=n} \operatorname{Tor}^{-i,j}_A(M,N)\Bigr)\Big/ F^{-p+1}\Bigl(\sum_{-i+j=n} \operatorname{Tor}^{-i,j}_A(M,N)\Bigr).$$

Topological applications of Theorem A.4.1 arise in the case when A, M, N are cochain algebras of topological spaces. The classical situation is described by the commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E_0 \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & B_0, \end{array} \qquad \text{(A.10)}$$

where $E_0 \to B_0$ is a Serre fibre bundle with fibre F over a simply connected base B_0, and $E \to B$ is the pullback along a continuous map $B \to B_0$. For any space X, let

$C^*(X)$ denote the singular $\mathbf{k}$-cochain algebra of X. Then $C^*(E_0)$ and $C^*(B)$ are $C^*(B_0)$-modules. Under these assumptions the following statement holds.

Lemma A.4.2 ([**177**, Proposition 3.4])**.** $\mathrm{Tor}_{C^*(B_0)}(C^*(E_0), C^*(B))$ *is a* $\mathbf{k}$*-algebra in a natural way, and there is a canonical isomorphism of algebras*

$$\mathrm{Tor}_{C^*(B_0)}\big(C^*(E_0), C^*(B)\big) \to H^*(E).$$

Applying Theorem A.4.1 in the case $A = C^*(B_0)$, $M = C^*(E_0)$, $N = C^*(B)$ and taking into account Lemma A.4.2, we come to the following statement.

Theorem A.4.3 (Eilenberg–Moore)**.** *There exists a spectral sequence* $\{E_r, d_r\}$ *of commutative algebras converging to* $H^*(E)$ *with*

$$E_2^{-i,j} = \mathrm{Tor}^{-i,j}_{H^*(B_0)}\big(H^*(E_0), H^*(B)\big).$$

The spectral sequence of Theorem A.4.3 is known as the (topological) *Eilenberg–Moore spectral sequence*. The case when B is a point is of particular importance, and we state the corresponding result separately.

Corollary A.4.4. *Let* $E \to B$ *be a fibration over a simply connected space* B *with fibre* F*. Then there exists a spectral sequence* $\{E_r, d_r\}$ *of commutative algebras with*

$$E_2 = \mathrm{Tor}_{H^*(B_0)}\big(H^*(E_0), \mathbf{k}\big).$$

We refer to the spectral sequence of Corollary A.4.4 as the *Eilenberg–Moore spectral sequence of fibration* $E \to B$. In the case when E_0 is a contractible space we obtain a spectral sequence converging to cohomology of the loop space ΩB_0.

Remark. In rational homotopy theory, the Sullivan–de Rham algebra $A^*(X)$ of piecewise polynomial forms is used as a (graded) commutative algebraic model for X, instead of the rational singular cochain algebra $C^*(X;\mathbb{Q})$, which is not commutative. It is proved in [**28**, §3] that the above results on the Eilenberg–Moore spectral sequence hold with C^* replaced by A^*. This result is not a direct corollary of algebraic properties of Tor, since the integration map $A^*(X) \to C^*(X, \mathbb{Q})$ is not multiplicative.

Bibliography

[1] J. Frank Adams. *On the cobar construction*. Proc. Nat. Acad. Sci. USA **42** (1956), 409–412.

[2] Ron M. Adin. *A new cubical h-vector*. Discrete Math. **157** (1996), no. 1–3, 3–14.

[3] Selman Akbulut and John D. McCarthy. *Casson's Invariant for Oriented Homology 3-spheres*. Math. Notes **36**. Princeton Univ. Press, Princeton, N.J., 1990.

[4] Michael F. Atiyah. *Bordism and cobordism*. Proc. Cambridge Philos. Soc. **57** (1961), 200–208.

[5] Michael F. Atiyah. *Convexity and commuting Hamiltonians*. Bull. London Math. Soc. **14** (1982), no. 1, 1–15.

[6] Michael F. Atiyah and Raoul Bott. *A Lefschetz fixed point formula for elliptic complexes. I.* Ann. of Math. **86** (1967), no. 2, 374–407.

[7] Michael F. Atiyah and Friedrich Hirzebruch. *Spin-manifolds and group actions*. In: *Essays in Topology and Related Topics*. Springer-Verlag, Berlin-Heidelberg-New York, 1970, pp. 18–28.

[8] Michèle Audin. *The Topology of Torus Actions on Symplectic Manifolds*. Progress in Mathematics **93**. Birkhäuser, Basel, 1991.

[9] Anton Ayzenberg and Victor Buchstaber. *Moment-angle complexes and polyhedral products for convex polytopes*. Preprint (2010); arXiv:1010.1922.

[10] Eric K. Babson, Louis J. Billera and Clara S. Chan. *Neighborly cubical spheres and a cubical lower bound conjecture*. Israel J. Math. **102** (1997), 297–315.

[11] Antony Bahri, Martin Bendersky, Frederic R. Cohen, and Samuel Gitler. *The polyhedral product functor: a method of computation for moment-angle complexes, arrangements and related spaces*. Adv. Math. **225** (2010), no. 3, 1634–1668.

[12] David Barnette. *Diagrams and Schlegel diagrams*. In: *Combinatorial Structures and Their Applications*. Proc. Calgary Internat. Conference, Calgary 1969. Gordon and Breach, New York, 1970, pp. 1–4.

[13] David Barnette. *The minimum number of vertices of a simple polytope*. Israel J. Math. **10** (1971), 121–125.

[14] David Barnette. *A proof of the lower bound conjecture for convex polytopes*. Pacific J. Math. **46** (1973), 349–354.

[15] David Barnette. *The triangulations of the 3-sphere with up to 8 vertices*. J. Combinatorial Theory. Ser. A **14** (1973), 37–52.

[16] Ilia V. Baskakov. *Cohomology of K-powers of spaces and the combinatorics of simplicial subdivisions*. Uspekhi Mat. Nauk **57** (2002), no. 5, 147–148 (Russian); Russian Math. Surveys **57** (2002), no. 5, 989–990 (English translation).

[17] Ilia V. Baskakov. *Massey triple products in the cohomology of moment-angle complexes*. Uspekhi Mat. Nauk **58** (2003), no. 5, 199–200 (Russian); Russian Math. Surveys **58** (2003), no. 5, 1039–1041 (English translation).

[18] Ilia V. Baskakov, Victor M. Buchstaber and Taras E. Panov. *Cellular cochain algebras and torus actions*. Uspekhi Mat. Nauk **59** (2004), no. 3, 159–160 (Russian); Russian Math. Surveys **59** (2004), no. 3, 562–563 (English translation).

[19] Victor V. Batyrev. *Quantum cohomology rings of toric manifolds*. In: *Journées de Géométrie Algébrique d'Orsay (Juillet 1992)*. Astérisque **218**. Sociéte Mathématique de France, Paris, 1993, pp. 9–34; arXiv:alg-geom/9310004.

[20] Louis J. Billera and Carl W. Lee. *Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes*. Bull. Amer. Math. Soc. **2** (1980), 181–185.

[21] Louis J. Billera and Carl W. Lee. *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes*. J. Comb. Theory, Ser. A **31** (1981), 237–255.

[22] Anders Björner and Frank Lutz. *Simplicial manifolds, bistellar flips and 16-vertex triangulation of the Poincaré homology 3-sphere*. Experiment. Math. **9** (2000), no. 2, 275–289.

[23] Anders Björner and Martin Tancer. *Note: Combinatorial Alexander duality—a short and elementary proof*. Discrete Comput. Geom. **42** (2009), no. 4, 586–593; arXiv:0710.1172.

[24] Gerd Blind and Roswitha Blind. *Triangle-free polytopes with few facets*. Arch. Math. (Basel) **58** (1992), no. 6, 599–605.

[25] Frédéric Bosio and Laurent Meersseman. *Real quadrics in $\mathbb{C}^n$, complex manifolds and convex polytopes*. Acta Math. **197** (2006), no. 1, 53–127.

[26] Raoul Bott and Hans Samelson. *Application of the theory of Morse to symmetric spaces*. Amer. J. of Math. **80** (1958), 964–1029.

[27] Raoul Bott and Clifford Taubes. *On the self-linking of knots. Topology and physics*. J. Math. Phys. **35** (1994), no. 10, 5247–5287.

[28] A. K. Bousfield and Victor K. Gugenheim. *On PL de Rham Theory and Rational Homotopy Type*. Memoirs of the Amer. Math. Soc. **9**, no. 179. Amer. Math. Soc., Providence, R.I., 1976.

[29] Glen E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York–London, 1972.

[30] Arne Brønsted. *An Introduction to Convex Polytopes*. Graduate Texts in Math. **90**, Springer-Verlag, New-York, 1983.

[31] Winfried Bruns and Joseph Gubeladze. *Combinatorial invariance of Stanley–Reisner rings*. Georgian Math. J. **3** (1996), no. 4, 315–318.

[32] Winfried Bruns and Jürgen Herzog. *Cohen–Macaulay Rings*, revised edition. Cambridge Studies in Adv. Math., vol. **39**, Cambridge Univ. Press, Cambridge, 1998.

[33] Victor M. Buchstaber [Bukhshtaber]. *The Chern–Dold character in cobordisms I*. Mat. Sbornik **83** (1970), no. 4, 575–595 (Russian); Math. USSR Sbornik **12** (1970), no. , 573–594 (English translation).

[34] Victor M. Buchstaber. *Ring of Simple Polytopes and Differential Equations*. Trudy Mat. Inst. Steklova **263** (2008), 18–43 (Russian); Proc. Steklov Inst. Math. **263** (2008), 1–25 (English translation).

[35] Victor M. Buchstaber. *Lectures on toric topology*. In *Proceedings of Toric Topology Workshop KAIST 2008*. Trends in Math. **10**, no. 1. Information Center for Mathematical Sciences, KAIST, 2008, pp. 1–64.

[36] Victor M. Buchstaber and Elena V. Koritskaya. *Quasilinear Burgers–Hopf equation and Stasheff polytopes*. Funktsional. Anal. i Prilozhen. **41** (2007), no. 3, 34–47 (Russian); Funct. Anal. and Appl. **41** (2007), no. 3, 196–207 (English translation).

[37] Victor M. Buchstaber and Nickolai Erokhovets. *Polytopes, Hopf algebras and quasi-symmetric functions*. Preprint (2010), arXiv:1011.1536.

[38] Victor M. Buchstaber and Tarea E. Panov. *Algebraic topology of manifolds defined by simple polyhedra*. Uspehi Mat. Nauk **53** (1998), no.3, 195–196 (Russian); Russian Math. Surveys **53** (1998), no. 3, 623–625 (English translation).

[39] Victor M. Buchstaber and Taras E. Panov. *Torus actions and the combinatorics of polytopes*. Trudy Mat. Inst. Steklova **225** (1999), 96–131 (Russian); Proc. Steklov Inst. Math. **225** (1999), 87–120 (English translation).

[40] Victor M. Buchstaber and Taras E. Panov. *Torus actions, combinatorial topology and homological algebra*. Uspekhi Mat. Nauk **55** (2000), no. 5, 3–106 (Russian). Russian Math. Surveys **55** (2000), no. 5, 825–921 (English translation).

[41] Victor M. Buchstaber and Taras E. Panov. *Torus Actions and Their Applications in Topology and Combinatorics*. University Lecture Series **24**. Amer. Math. Soc., Providence, R.I., 2002.

[42] Victor M. Buchstaber and Taras E. Panov. *Combinatorics of simplicial cell complexes and torus action*. Trudy Mat. Inst. Steklova **247** (2004), pp. 41–58 (Russian); Proc. Steklov Inst. Math., vol. **247** (2004), pp. 33–49 (English translation).

[43] Victor M. Buchstaber and Taras E. Panov. *Torus actions in topology and combinatorics* (in Russian). MCCME, Moscow, 2004.

[44] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Spaces of polytopes and cobordism of quasitoric manifolds*. Moscow Math. J. **7** (2007), no. 2, 219–242; arXiv:math.AT/0609346.

[45] Victor M. Buchstaber, Taras E. Panov and Nigel Ray. *Toric genera*. Internat. Math. Research Notices **16** (2010), 3207–3262; arXiv:0908.3298.

[46] Victor M. Buchstaber [Bukhshtaber] and Nigel Ray. *Toric manifolds and complex cobordisms*. Uspekhi Mat. Nauk **53** (1998), no. 2, 139–140 (Russian); Russian Math. Surveys **53** (1998), no. 2, 371–373 (English translation).

[47] Victor M. Buchstaber and Nigel Ray. *Flag manifolds and the Landweber–Novikov algebra*. Geom. Topol. **2** (1998), 79–101; arXiv:math.AT/9806168.

[48] Victor M. Buchstaber and Nigel Ray. *Tangential structures on toric manifolds, and connected sums of polytopes*. Internat. Math. Res. Notices **4** (2001), 193–219; arXiv:math.AT/0010025.

[49] Victor M. Buchstaber and Nigel Ray. *An invitation to toric topology: vertex four of a remarkable tetrahedron*. In: *Toric Topology* (M. Harada *et al.*, eds.). Contemp. Math., vol. **460**, Amer. Math. Soc., Providence, RI, 2008, pp. 1–27.

[50] Victor M. Buchstaber, Vadim Volodin. *Upper and lower bound theorems for graph-associahedra*. Izv. Ross. Akad. Nauk, Ser. Mat. **74** (2010), to appear; Izv. Math. **74** (2010), to appear (English translation); arXiv:1005.1631.

[51] James W. Cannon. *Shrinking cell-like decompositions of manifolds. Codimension three.* Ann. of Math. (2) **110** (1979), no. 1, 83–112.

[52] Michael P. Carr and Satyan L. Devadoss. *Coxeter complexes and graph-associahedra.* Topology Appl. **153** (2006), no. 12, 2155–2168; arXiv:math.QA/0407229.

[53] Ruth Charney and Michael Davis. *The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold.* Pacific J. Math. **171** (1995), no. 1, 117–137.

[54] Suyoung Choi, Mikiya Masuda and Dong Youp Suh. *Quasi-toric manifolds over a product of simplices.* Preprint (2008); arXiv:0803.2749.

[55] Yusuf Civan and Nigel Ray. *Homotopy decompositions and K-theory of Bott towers.* K-Theory **34** (2005), no. 1, 1–33.

[56] George F. Clements and and Bernt Lindström. *A generalization of a combinatorial theorem of Macaulay.* J. Combinatorial Theory **7** (1969), 230–238.

[57] Pierre E. Conner and Edwin E. Floyd. *Differentiable periodic maps.* Academic Press Inc., New York, 1964.

[58] Pierre E. Conner and Edwin E. Floyd. *The Relation of Cobordism to K-theories.* Lecture Notes in Math. **28**, Springer-Verlag, Berlin, 1966.

[59] David A. Cox. *The homogeneous coordinate ring of a toric variety.* J. Algebraic Geom. **4** (1995), no. 1, 17–50; arXiv:alg-geom/9210008.

[60] David A. Cox. *Recent developments in toric geometry.* In: *Algebraic geometry—Santa Cruz 1995.* Proc. Sympos. Pure Math. **62**, Part 2. AMS, Providence, R.I., 1997, pp. 389–436; arXiv:alg-geom/9606016.

[61] H. S. M. Coxeter. *Regular Polytopes.* 3rd edition. Dover Publications, 1973.

[62] Vladimir I. Danilov *The geometry of toric varieties.* Uspekhi Mat. Nauk **33** (1978), no. 2, 85–134 (Russian); Russian Math. Surveys **33** (1978), no. 2, 97–154 (English translation).

[63] Michael W. Davis. *Groups generated by reflections and aspherical manifolds not covered by Euclidean space.* Ann. of Math. **117** (1983), no. 2, 293–324.

[64] Michael W. Davis and Tadeusz Januszkiewicz. *Convex polytopes, Coxeter orbifolds and torus actions.* Duke Math. J., **62** (1991), no. 2, 417–451.

[65] Corrado De Concini and C. Procesi. *Wonderful models of subspace arrangements.* Selecta Math. (N.S.) **1** (1995), no. 3, 459–494.

[66] M. Dehn. *Die Eulersche Formel im Zusammenhang mit dem Inhalt in der nicht-Euklidischen Geometrie.* Math. Ann. **61** (1905), 561–568.

[67] Mark de Longueville. *The ring structure on the cohomology of coordinate subspace arrangements.* Math. Z. **233** (2000), no. 3, 553–577.

[68] Thomas Delzant. *Hamiltoniens périodiques et images convexes de l'application moment.* Bull. Soc. Math. France **116** (1988), no. 3, 315–339.

[69] Graham Denham and Alexander I. Suciu. *Moment-angle complexes, monomial ideals, and Massey products.* Pure Appl. Math. Q. **3** (2007), no. 1, 25–60; arXiv:math.AT/0512497.

[70] Natalia Dobrinskaya. *The classification problem for quasitoric manifolds over a given simple polytope.* Funktsional. Anal. i Prilozhen. **35** (2001), no. 2, 3–11 (Russian); Funct. Anal. and Appl. **35** (2001), no. 2, 83–89 (English translation).

[71] Nikolai P. Dolbilin, Mikhail A. Shtan'ko and Mikhail I. Shtogin. *Cubic manifolds in lattices.* Izv. Ross. Akad. Nauk Ser. Mat. **58** (1994), no. 2, 93–107 (Russian); Russian Acad. Sci. Izv. Math. **44** (1995), no. 2, 301–313 (English translation).

[72] Albrecht Dold. *Geometric cobordism and the fixed point transfer.* In: *Algebraic Topology, Proceedings Vancouver 1977.* Lecture Notes in Math. **673**. Springer Verlag, Berlin, 1978, pp. 32–87.

[73] Johannes J. Duistermaat and G. J. Heckman. *On the variation in the cohomology of the symplectic form of the reduced phase space.* Invent. Math. **69** (1982), no. 2, 259–268.

[74] Art M. Duval. *Free resolutions of simplicial posets.* J. Algebra **188** (1997), 363–399.

[75] Robert D. Edwards. *The double suspension of a certain homology 3-sphere is* S^5. Notices AMS **22**, A-334 (1975).

[76] Robert D. Edwards. *The topology of manifolds and cell-like maps.* In: Proceedings of the International Congress of Mathematicians (Helsinki, 1978). Acad. Sci. Fennica, Helsinki, 1980, pp. 111–127.

[77] Samuel Eilenberg and John C. Moore. *Homology and fibrations. I.* Comment. Math. Helv. **40** (1966), 199–236.

[78] Davin Eisenbun. *Commutative Algebra with a View Toward Algebraic Geometry.* Graduate Texts in Math. **150**, Springer-Verlag, New-York, 1995.

[79] Günter Ewald. *Combinatorial Convexity and Algebraic Geometry.* Graduate Texts in Math. **168**, Springer-Verlag, New-York, 1996.

[80] Günter Ewald and Christoph Schulz. *Nonstarshaped spheres.* Arch. Math. (Basel) **59** (1992), no. 4, 412–416.

[81] Eva Maria Feichtner and Bernd Sturmfels. *Matroid polytopes, nested sets and Bergman fans.* Port. Math. (N.S.) **62** (2005), no. 4, 437–468; arXiv:math.CO/0411260.

[82] John Fogarty, Frances Kirwan and David Mumford. *Geometric Invariant Theory.* Third Edition. Springer-Verlag, Berlin, 1994.

[83] Sergey Fomin and Andrei Zelevinsky. *Cluster algebras. I. Foundations.* J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[84] Matthias Franz. *On the integral cohomology of smooth toric varieties.* Proc. Steklov Inst. Math. **252** (2006), 53–62; arXiv:math.AT/0308253.

[85] Michael H. Freedman and Frank Quinn. *Topology of 4-manifolds.* Princeton Mathematical Series **39**. Princeton Univ. Press, Princeton, N.J., 1990.

[86] William Fulton. *Introduction to Toric Varieties.* Ann. of Math. Studies **131**. Princeton Univ. Press, Princeton, N.J., 1993.

[87] Światosław R. Gal. *Real root conjecture fails for five and higher dimensional spheres.* Discrete Comput. Geom. **34** (2005), no. 2, 260–284; arXiv:math/0501046.

[88] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants.* Birkhäuser, Boston, 1994.

[89] Samuel Gitler and Santiago López de Medrano. *Intersections of quadrics, moment-angle manifolds and connected sums.* Preprint (2009); arXiv:0901.2580.

[90] Mark Goresky, R. Kottwitz and Robert MacPherson. *Equivariant cohomology, Koszul duality and the localisation theorem.* Invent. Math. **131** (1998), no. 1, 25–83.

[91] Mark Goresky and Robert MacPherson. *Stratified Morse Theory.* Springer-Verlag, Berlin–New York, 1988.

[92] Jelena Grbić and Stephen Theriault. *The homotopy type of the complement of the codimension-two coordinate subspace arrangement. Uspekhi Mat. Nauk,* 59(6):203–204, 2004 (Russian); *Russian Math. Surveys,* 59(3):1207–1209, 2004 (English).

[93] Jelena Grbić and Stephen Theriault. *The homotopy type of the complement of a coordinate subspace arrangement.* Topology **46** (2007), no. 4, 357–396; arXiv:math/0601279.

[94] Mikhail Gromov. *Hyperbolic groups.* In: *Essays in Group Theory*, S. M. Gersten, ed. MSRI Publ. **8**. Springer-Verlag, New York, 1987, pp. 75–263.

[95] Michael Grossberg and Yael Karshon. *Bott towers, complete integrability, and the extended character of representations.* Duke Math. J. **76** (1994), 23–58.

[96] Branko Grünbaum. *Convex Polytopes.* Second edition. Graduate Texts in Math. **221**. Springer-Verlag, New York, 2003.

[97] Branko Grünbaum and V. P. Sreedharan. *An enumeration of simplicial 4-polytopes with 8 vertices.* J. Combinatorial Theory **2** (1967), 437–465.

[98] Victor Guillemin. *Moment Maps and Combinatorial Invariants of Hamiltonian* T^n*-spaces.* Progress in Math. **122**. Birkhaüser, Boston, 1994.

[99] Victor Guillemin and Shlomo Sternberg. *Convexity properties of the moment mapping.* Inv. Math. **67** (1982), no. 3, 491–513.

[100] Victor Guillemin and Shlomo Sternberg. *Supersymmetry and Equivariant de Rham Theory*. Springer-Verlag, Berlin-Heidelberg-New York, 1999.

[101] Victor W. Guillemin and Catalin Zara. *Equivariant de Rham theory and graphs*. Asian J. Math. **3** (1999), no. 1, 49–76; arXiv:math.DG/9808135.

[102] Victor W. Guillemin and Catalin Zara. *One-skeleta, Betti numbers and equivariant cohomology*. Duke Math. J. **107** (2001), no. 2, 283–349; arXiv:math/9903051.

[103] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, Cambridge, 2002.

[104] Akio Hattori. *Symplectic manifolds with semi-free Hamiltonian S^1-action*. Tokyo J. Math. **15** (1992), no. 2, 281–296.

[105] Akio Hattori and Mikiya Masuda, *Theory of multi-fans*. Osaka J. Math. **40** (2003), 1–68; arXiv:math/0106229.

[106] Akio Hattori and Mikiya Masuda, *Elliptic genera, torus manifolds and multi-fans*. Internat. J. of Math. **16** (2005), no. 9, 957–998; arXiv:math/0107014.

[107] Akio Hattori and Tomoyoshi Yoshida. *Lifting compact group actions in fiber bundles*. Japan. J. Math. **2** (1976), 13–25.

[108] Friedrich Hirzebruch. *Komplexe Mannigfaltigkeiten.* (German) Proc. Internat. Congress Math, 1958. Cambridge Univ. Press, New York, 1960, pp. 119–136.

[109] Friedrich Hirzebruch. *Topological methods in algebraic geometry*, 3rd edition. Springer-Verlag, Berlin–Heidelberg, 1966.

[110] Friedrich Hirzebruch, Thomas Berger and Rainer Jung. *Manifolds and Modular Forms*, Second Edition. A Publication from the Max-Planck-Institut für Mathematik, Bonn. Friedr. Vieweg & Sohn, Braunschweig, 1994.

[111] Melvin Hochster. *Cohen–Macaulay rings, combinatorics, and simplicial complexes.* In: *Ring Theory II (Proc. Second Oklahoma Conference)*. B. R. McDonald and R. Morris, eds. Dekker, New York, 1977, pp. 171–223.

[112] J. F P. Hudson. *Piecewise Linear Topology*. W. A. Benjamin, Inc., New York–Amsterdam, 1969.

[113] Dmitri Ilinskii. *On the almost free action of one-dimensional torus on a toric variety*. Mat. Sbornik **197** (2006), no. 5, 51–74 (Russian); Sbornik Math. **197** (2006), no. 5, 681–703 (English translation).

[114] Gil Kalai. *Rigidity and the lower bound Theorem. I.* Inv. Math. **88** (1987), no. 1, 125–151.

[115] Askold G. Khovanskii [Hovanskii]. *Newton polyhedra and toric varieties.* Funk. Anal. i Priloz **11** (1977), no. 4, 56–64 (Russian); Funct. Anal. Appl. **11** (1977), no. 4, 289–296 (English translation).

[116] Askold G. Khovanskii. *Hyperplane sections of polyhedra, toric varieties and discrete groups in Lobachevskii space*. Funk. Anal. i Prilozhen. **20** (1986), no. 1, 50–61 (Russian); Funct. Anal. Appl. **20** (1986), no. 1, 41–50 (English translation).

[117] Robion C. Kirby and Laurent C. Siebenmann. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*. Ann. of Math. Stud. **88**, Princeton Univ. Press, Princeton, N.J., 1977.

[118] Victor Klee. *A combinatorial analogue of Poincaré's duality theorem*. Canad. J. Math. **16** (1964), 517–531.

[119] Victor Klee and Peter Kleinschmidt, *Convex polytopes and related complexes*. In *Handbook of Combinatorics, Vol. 1,2*. R. L. Graham et al, eds. Elsevier Science B.V., Amsterdam, 1995, pp. 875–917.

[120] Igor M. Krichever. *Formal groups and the Atiyah-Hirzebruch formula*. Izv. Akad. Nauk SSSR **38** (1974), no. 6, 1289–1304 (Russian); Math. USSR Izv. **8** (1974), no. 6, 1271–1285 (English translation).

[121] Michel Lazard. *Sur le groupes de Lie formels à un paramètre*. Bull. Soc. Math. France **83** (1955), no. 3, 251–274.

[122] Peter S. Landweber. *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*. Amer. J. Math. **98** (1976), no. 3, 591–610.

[123] Zhi Lü and Taras Panov. *Moment-angle complexes from simplicial posets*. Cent. Eur. J. Math. **9** (2011), no. 4, 715–730; arXiv:0912.2219.

[124] Frank H. Lutz. *Triangulated manifolds with few vertices and vertex-transitive group actions.* Dissertation, Technischen Universität Berlin, Berlin, 1999. Berichte aus der Mathematik, Shaker Verlag, Aachen, 1999.

[125] F. S. Macaulay. *Some properties of enumeration in the theory of modular systems.* Proc. London Math. Soc. **26** (1927), 531–555.

[126] Saunders Maclane. *Homology.* Springer-Verlag, Berlin, 1963.

[127] Hiroshi Maeda, Mikiya Masuda and Taras Panov. *Torus graphs and simplicial posets.* Advances in Math. **212** (2007), no. 2, 458–483; arXiv:math/0511582.

[128] Peter Mani. *Spheres with few vertices.* J. Combinatorial Theory. Ser. A **13** (1972), 346–352.

[129] Mikiya Masuda. *Unitary toric manifolds, multi-fans and equivariant index.* Tohoku Math. J. **51** (1999), no. 2, 237–265.

[130] Mikiya Masuda. *h-vectors of Gorenstein* simplicial posets.* Adv. Math. **194** (2005), no. 2, 332–344; arXiv:math/0305203.

[131] Mikiya Masuda and Taras Panov. *On the cohomology of torus manifolds.* Osaka J. Math. **43** (2006), 711–746; arXiv:math/0306100.

[132] Mikiya Masuda and Taras Panov. *Semifree circle actions, Bott towers, and quasitoric manifolds.* Mat. Sbornik **199** (2008), no. 8, 95–122 (Russian); Sbornik Math. **199** (2008), no. 7–8, 1201–1223 (English translation); arXiv:math.AT/0607094.

[133] Sergei V. Matveev. *Algorithmic topology and classification of 3-manifolds.* Algorithms and Computation in Mathematics, **9**. Springer-Verlag, Berlin, 2003.

[134] John McCleary. *A User's Guide to Spectral Sequences,* 2nd edition. Cambdridge Univ. Press, Cambridge, 2001.

[135] Peter McMullen. *The maximum numbers of faces of a convex polytope.* Mathematika **17** (1970), 179–184.

[136] Peter McMullen. *The numbers of faces of simplicial polytopes.* Israel J. Math. **9** (1971), 559–570.

[137] Peter McMullen. *On simple polytopes.* Invent. Math. **113** (1993), no. 2, 419–444.

[138] Peter McMullen. *Weights on polytopes.* Discrete Comput. Geom. **15** (1996), no. 4, 363–388.

[139] Peter McMullen and G. Shephard. *Convex Polytopes and the Upper Bound Conjecture.* London Math. Soc. Lecture Notes Series **3**, Cambridge Univ. Press, Cambridge, 1971.

[140] Peter McMullen and David Walkup. *A generalized lower bound conjecture for simplicial polytopes.* Mathematika **18** (1971), 264–273.

[141] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra.* Graduate Texts in Math. **227**. Springer-Verlag, New York, 2005.

[142] John Milnor. *On the cobordism ring Ω^* and a complex analogue. I.* Amer. J. Math. **82** (1960), 505–521.

[143] Andrey Mironov and Taras Panov. *Intersections of quadrics, moment-angle manifolds, and Hamiltonian-minimal Lagrangian embeddings.* Preprint (2011); arXiv:1103.4970.

[144] Edwin E. Moise. *Geometric Topology in Dimensions* 2 *and* 3. Graduate Texts in Math. **47**. Springer-Verlag, New York-Heidelberg, 1977.

[145] James R. Munkres. *Elementary differential topology.* Revised edition. Annals of Math. Studies **54**. Princeton Univ. Press, Princeton, N.J., 1966.

[146] Oleg R. Musin. *Actions of the circle on homotopy complex projective spaces.* Mat. Zametki **28** (1980), no. 1, 139–152 (Russian).

[147] A. Nabutovsky. *Geometry of the space of triangulations of a compact manifold.* Comm. Math. Phys. **181** (1996), no. 2, 303–330.

[148] Dietrich Notbohm and Nigel Ray. *On Davis–Januszkiewicz homotopy types I; formality and rationalisation.* Algebraic & Geometric Topology **5** (2005), 31–51; arXiv:math/0311167.

[149] Sergei P. Novikov. *Some problems in the topology of manifolds connected with the theory of Thom spaces.* Dokl. Akad. Nauk SSSR **132** (1960), 1031–1034 (Russian). Soviet Math. Dokl. **1** (1960), 717–720 (English translation).

[150] Sergei P. Novikov. *Homotopy properties of Thom complexes.* Mat. Sbornik **57(99)** (1962), 407–442 (Russian).

[151] Sergei P. Novikov. *Methods of algebraic topology from the point of view of cobordism theory.* Izv. Akad. Nauk SSSR, Ser. Mat. **31** (1967), no. 4, 855–951 (Russian). Math. USSR – Izv. **1** (1967), no. 4, 827–913 (English translation).

[152] Sergei P. Novikov. *Topology I.* Encyclopaedia Math. Sci. **12**. Springer-Verlag, Berlin, 1996.

[153] Tadao Oda. *Convex Bodies and Algebraic Geometry. An introduction to the theory of toric varieties.* Springer-Verlag, New York, 1988.

[154] Udo Pachner. *Konstruktionsmethoden und das kombinatorische Homöomorphieproblem für Triangulationen kompakter semilinearer Mannigfaltigkeiten.* Abh. Math. Sem. Univ. Hamburg **57** (1987), 69–86.

[155] Udo Pachner. *P.L. homeomorphic manifolds are equivalent by elementary shellings.* European J. Combin. **12** (1991), no. 2, 129–145.

[156] Taras E. Panov. *Classification up to cobordism of manifolds that carry a simple action of the group* $\mathbb{Z}/p$. Mat. Zametki **63** (1998), no. 2, 260–268 (Russian). Math. Notes **63** (1998), no. 1–2, 225–232 (English translation); arXiv:math.AT/9908166.

[157] Taras E. Panov. *Calculation of Hirzebruch genera of manifolds with* $\mathbb{Z}/p$ *group action in terms of the invariants of the action.* Izv. Ross. Acad. Nauk, Ser. Mat. **62** (1998), no. 3, 87–120 (Russian); Izv. Math. **62** (1998), no. 3, 515–548 (English translation); arXiv:math.AT/9909081.

[158] Taras E. Panov. *Hirzebruch genera of manifolds with torus actions.* Izv. Ross. Akad. Nauk, Ser. Mat. **65** (2001), no. 3, 123–138 (Russian); Izv. Math. **65** (2001), no. 3, 543–556 (English translation); arXiv: math.AT/9910083.

[159] Taras Panov. *Cohomology of face rings, and torus actions.* In: *Surveys in Contemporary Mathematics.* London Math. Soc. Lecture Note Series, vol. **347**, Cambridge, U.K., 2008, pp. 165–201; arXiv:math.AT/0506526.

[160] Taras Panov. *Toric Kempf–Ness sets.* Trudy Mat. Inst. Steklova **263** (2008), 159–172 (Russian); Proc. Steklov Inst. Math. **263** (2008), pp. 150–162 (English translation); arXiv:math.AG/0603556.

[161] Taras Panov. *Moment-angle manifolds and complexes.* Trends in Mathematics - New Series. Information Center for Mathematical Sciences, KAIST. Vol. 12 (2010), no. 1, pp. 43–69; arXiv:1008.5047.

[162] Taras Panov and Nigel Ray. *Categorical aspects of toric topology.* In: *Toric Topology*, M. Harada *et al.*, eds. Contemp. Math., vol. **460**. Amer. Math. Soc., Providence, RI, 2008, pp. 293–322; arXiv:0707.0300.

[163] Taras Panov, Nigel Ray and Rainer Vogt. *Colimits, Stanley–Reiner algebras, and loop spaces.* In: *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001).* Progress in Math., vol. **215**, Birkhäuser, Basel, 2004, pp. 261–291; arXiv:math/0202081.

[164] Taras Panov and Yuri Ustinovsky. *Complex-analytic structures on moment-angle manifolds.* Moscow Math. J. **12** (2012); arXiv:1008.4764.

[165] Lev S. Pontryagin, *Foundations of Combinatorial Topology.* Graylock Press, Rochester, N.Y., 1952.

[166] Alexander Postnikov. *Permutohedra, associahedra, and beyond.* Int. Math. Res. Not. **2009**, no. 6, 1026–1106; arXiv:math.CO/0507163.

[167] Alexander Postnikov, Victor Reiner, L. Williams. *Faces of generalized permutohedra.* Doc. Math. **13** (2008), 207–273; arXiv:math.CO/0609184.

[168] Daniel G. Quillen. *On the formal group laws of unoriented and complex cobordism theory.* Bull. Amer. Math. Soc. **75** (1969), no. 6, 1293–1298.

[169] Daniel G. Quillen. *Elementary proofs of some results of cobordism theory using steenrod operations. Adv. Math.* **7** (1971), 29–56.

[170] Andrew Ranicki. *On the Hauptvermutung.* In: *The Hauptvermutung Book.* K-Monogr. Math. **1**, Kluwer Acad. Publ., Dordrecht, 1996, pp. 3–31.

[171] Nigel Ray. *On a construction in bordism theory. Proc. Edin. Math. Soc.* **29** (1986), 413–422.

[172] Gerald A. Reisner. *Cohen–Macaulay quotients of polynomial rings*, Adv. Math. **21** (1976), no. 1, 30–49.

[173] Colin P. Rourke and Brian J. Sanderson. *Introduction to Piecewise-Linear Topology.* Springer-Verlag, Berlin, 1972.

[174] J. Hyam Rubinstein. *An algorithm to recognise the 3-sphere.* Proceedings of the International Congress of Mathematicians, Vol 1, 2 (Zürich, 1994), 601-–611; Birkhäuser, Basel, 1995.

[175] Yuli B. Rudyak. *Piecewise linear structures on topological manifolds.* arXiv:math.AT/0105047 (2001).

[176] Gerald W. Schwarz. *The topology of algebraic quotients.* In: *Topological methods in algebraic transformation groups.* Progr. in Math **80**. Birkhäuser, Boston, 1989, pp. 135–151.

[177] Larry Smith. *Homological algebra and the Eilenberg–Moore spectral sequence.* Trans. Amer. Math. Soc. **129** (1967), 58–93.

[178] D. M. Y. Sommerville. *The relations connecting the angle-sums and volume of a polytope in space of n dimensions.* Proc. Roy. Soc. London Ser. A **115** (1927), 103–119.

[179] Richard P. Stanley. *The upper bound conjecture and Cohen-Macaulay rings.* Studies in Appl. Math. **54** (1975), no. 2, 135–142.

[180] Richard P. Stanley. *The number of faces of simplicial convex polytope.* Adv. Math. **35** (1980), no. 3, 236–238.

[181] Richard P. Stanley. *Enumerative combinatorics*, Vol. 1. Wadsworth and Brooks/Cole, Monterey, California, 1986.

[182] Richard P. Stanley. *f-vectors and h-vectors of simplicial posets.* J. Pure Appl. Algebra. **71** (1991), 319–331.

[183] Richard P. Stanley. *A monotonicity property of h-vectors and h^*-vectors.* European J. Combin. **14** (1993), no. 3, 251–258.

[184] Richard P. Stanley. *Combinatorics and Commutative Algebra*, second edition. Progr. in Math. **41**. Birkhäuser, Boston, 1996.

[185] Richard P. Stanley. *Positivity problems and conjectures in algebraic combinatorics.* In: *Mathematics: Frontiers and Perspectives*, V. I. Arnold et al, eds. Amer. Math. Soc., Providence, R.I., 2000, pp. 295–319.

[186] James D. Stasheff. *Homotopy associativity of H-spaces. I.* Transactions Amer. Math. Soc. **108** (1963), 275–292.

[187] James D. Stasheff. *From operads to "physically" inspired theories.* In: *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995).* Contemp. Math. **202** (1997), 53–81.

[188] Robert E. Stong. *Notes on Cobordism Theory.* Princeton Univ. Press, Princeton, 1968.

[189] R. H. Szczarba. *On tangent bundles of fibre spaces and quotient spaces.* Amer. J. Math. **86** (1964), 685–697.

[190] René Thom. *Quelques propriétés globales des variétés différentiables.* Comment. Math. Helv. **28** (1954), 17–86.

[191] Emery Thomas. *Complex structures on real vector bundles.* Amer. J. Math. **89** (1967), 887–908.

[192] Vladlen Timorin. *An analogue of the Hodge–Riemann relations for simple convex polytopes.* Uspekhi Mat. Nauk **54** (1999), no. 2, 113–162 (Russian); Russian Math. Surveys **54** (1999), no. 2, 381–426 (English translation).

[193] Susan Tolman and Jonathan Weitsman. *On semifree symplectic circle actions with isolated fixed points.* Topology **39** (2000), no. 2, 299–309.

[194] Ernest B. Vinberg. *Discrete linear groups generated by reflections.* Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1072–1112 (Russian); Math. USSR Izvestija **5** (1971), no. 5, 1083–1119 (English translation).

[195] Ernest B. Vinberg and Vladimir L. Popov. *Invariant theory.* In: *Algebraic Geometry 4.* VINITI, Moscow, 1989, pp. 137–314 (Russian); Encyclopaedia Math. Sci. **55** Springer, Heidelberg, 1994, pp. 123–284 (English translation).

[196] I. A. Volodin, V. E. Kuznecov, A. T. Fomenko. *The problem of the algorithmic discrimination of the standard three-dimensional sphere.* Appendix by S. P. Novikov. Uspehi Mat. Nauk **29** (1974), no. 5, 71–168 (Russian); Russian Math. Surveys **29** (1974), no. 5, 71–172 (English translation).

[197] V. A. Yemelichev, M. M. Kovalev and M. K. Kravtsov. *Polytopes, Graphs and Optimisation.* Translated from the Russian by G. H. Lawden. Cambridge Univ. Press, Cambridge, 1984.

[198] C. T. C. Wall. *Determination of the cobordism ring.* Ann. of Math. (2) **72** (1960), 292-–311.

[199] Günter M. Ziegler. *Lectures on polytopes.* Graduate Texts in Mathematics, vol. **152**, Springer-Verlag, New York, 1995.